Introduction to Reaction-Advection-Diffusion Equations

Oceanic ecosystems are difficult to understand not only because of the complex nature of the biological interactions and of the physical flows, but also because the interplay between these two kinds of processes can be significant. A practical model – meaning one which has a structure that can be defended as plausible and which is computationally feasible – inevitably represents an enormous simplification of the system. In most cases, models are expressed as reaction-advection-diffusion equations; this chapter will deal with a very simple version of such a model in order to explicate the basic steps which go into its construction, analysis, and solution. Keep in mind, throughout, that we are building a "toy model" to illustrate the general (and common) approach. We shall pass over, but point out, many of the more difficult issues to be addressed in later chapters; issues which, we believe, must be confronted to improve our understanding of the combined physics and biology of the oceans.

The chapter begins with a consideration of the reaction terms – the ways in which different classes of organisms interact with each other and with the environment. In effect, we are isolating a bit of ocean in a container and trying to represent how it changes with time. Next, we think about a virtual box (defined by the latitudes, longitudes, and depths of the corners); properties in the box will change as water with different characteristics enters and exits. These advection and diffusion processes are usually associated with ocean physics, so that we must also consider how the flow structure is forced and shaped. A model may be available (perhaps one which can assimilate field data), or we may need to build it. Before taking this step, however, we can gain useful insights into the likely biological effects of the flows by using idealized velocities presumed to represent (perhaps roughly) the important parts of the actual or modelled currents. Then we are able to move to a fully coupled (or, perhaps more accurately, combined) physical-biological model and to see whether some of the more complicated details in flows have significant effects or not. This progression – from separate biological and physical models studied in isolation to idealized experiments and to a fully coupled calculation – offers a systematic approach that can increase our understanding of the system and can give us confidence in the correctness of our computations.

1.1 — Preliminaries

Most models of biological/physical processes in the ocean build upon four fundamental processes

- advection: movement by the ocean currents or by the organisms
- diffusion: mixing processes, whether by molecular motion, turbulence, or movement of organisms/ material relative to the water
- reaction: processes which transfer energy/ biomass/ carbon/... from one group of organisms (or other categories such as dissolved CO_2 or non-living particulates) to another. These are generally framed as though they were kinds of chemical reactions; the rates at which the concentration of one component changes is a function of the concentrations of this property and of its resources or prey and its consumers or predators.

- physical forcing: winds, fluxes of heat or freshwater, and tides which generate flows and alter the temperature and salinity structure.

The first three of these lead to the basic PDE's governing the interior dynamics. These are the reaction-advection-diffusion equations

$$\frac{\partial}{\partial t} b_i + \nabla \cdot (\mathbf{u} b_i) = \nabla \kappa \nabla b_i + \mathcal{B}_i(b_1, b_2, b_3, \ldots)$$

where $b_i(\mathbf{x}, t)$ is the density of the i^{th} class of organisms, $\mathcal{B}_i$ represents the processes which transfer biomass or carbon/ nitrogen mass to this class from the others, κ is the diffusivity, and $\mathbf{u}$ is the advecting velocity.†

We shall address all of these in much more detail, although one might already anticipate that biological processes will turn out to be much more complicated than chemical reactions. However, it is useful to begin with a simple example and a simplified derivation to introduce some of the basic ideas behind such models. We shall explore the dynamics of phytoplankton [modelled as a density $P(\mathbf{x}, t)$], zooplankton [$Z(\mathbf{x}, t)$], and nutrients [$N(\mathbf{x}, t)$] in an upwelling system (figure 1.1). We would like

We shall consider the effects of a wind blowing along-shore. As Ekman realized, the transport in the near surface is 90° to the right of the wind (in the northern hemisphere) – see 1.xx. Essentially, on scales comparable to the mixed-layer depth, the pressure forces and advection of momentum are small, so that the force applied by the wind is balanced by Coriolis forces associated with the offshore flow:

$$f \hat{\mathbf{z}} \times \int_{-h}^{0} \langle \mathbf{u} \rangle = \frac{\vec{\tau}_0}{\rho} \quad \text{or} \quad \int_{-h}^{0} \langle \mathbf{u} \rangle = -\hat{\mathbf{z}} \times \frac{\vec{\tau}_0}{\rho f}$$

where τ_0 is the stress on the sea surface imparted by the wind (in force per area units†) and h is the depth of the surface layer. A southward wind on a western coast (Oregon, California...) gives off-shore transport in the surface layer, which must be compensated for by deeper on-shore flows.

The resulting flow begins to lift the density surfaces near the coast; in turn, the isopycnals tend to slump back towards level and begin to counteract the offshore tendency from the wind.

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† We shall use boldface such as $\mathbf{x}$ to indicate vectors in ordinary three-dimensional space, with indices or special variable names used as convenient. Thus $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ gives the east, north, and upward distances. Velocities $\mathbf{u} = (u, v, w)$ are denoted similarly.

† τ_0/ρ has the units of velocity squared.

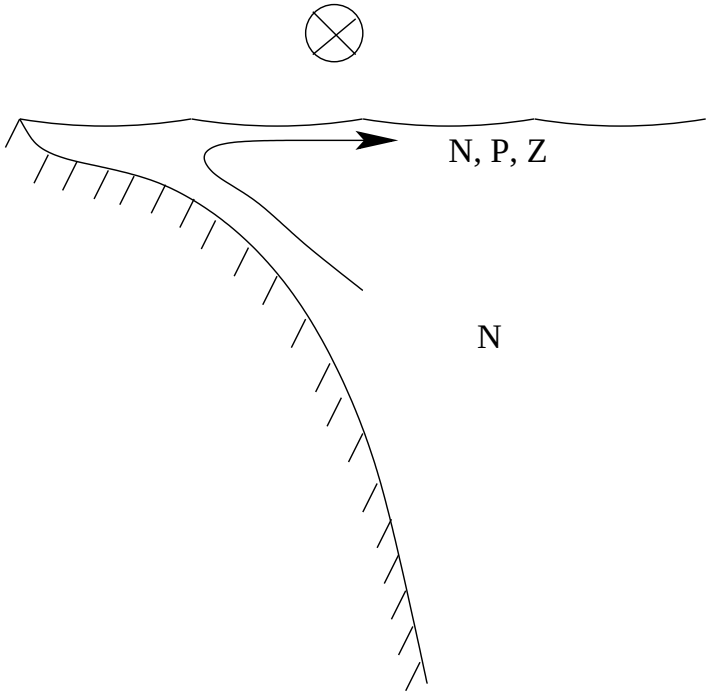

Figure 1.1: Sketch of upwelling system, showing along-shore wind, onshore deep flows, offshore surface flows, and upwelling near the coast. The phytoplankton and zooplankton live in the surface layers; nutrients are brought up in the upwelled water.

1.2 — Continuum Representations

To begin, we must consider how we can interpret a biological variable such as $P(\mathbf{x}, t)$ which will be used in a model. We intend it to represent the concentration of phytoplankton, but what does that mean when we are dealing with an assemblage of individual organisms? To what degree should we be concerned about the differences among them – in species, in physiological or physical state, etc.? Describing any single organism in detail can require a tremendous amount of information, and attempting to either model or measure all of it for every organism present would be an impossible task. Instead, we must adopt the census-taker's point of view: choose a limited set of data considered to be important and tabulate that for each member of the population. In the Warm Core Rings Program, for example, Wiebe, *et al.*(priv. comm.) recorded for each euphausiid its species, position, carapace length, wet weight, and lipid content (when possible).

Census data of this kind is not usually published; instead, the researchers present averages such as the mean wet weight for a particular species, the mean numbers per unit volume, or the biomass per unit volume in different depth ranges (c.f., Wiebe, *et al.*, 19xx). The latter calculation gives an estimate of the euphausiid density, $b_{euph}(\mathbf{x}, t)$. When we use this type of information, we are beginning to think of the biota as a **continuum** or what physicists call a **field**: the density is considered to be a continuous function of space (as well as time) much like temperature or salinity. The data is presumed to represent a few samples of this function.

When might we expect that a continuum representation could apply to organisms which are, after all, discrete? Let's start with an "experimental" point of view: can we define a number density $n(\mathbf{x}, t)$ which is a reasonably smooth (i.e., differentiable) function of position $\mathbf{x}$? Observationally, we would do this by counting numbers in a small volume V around $\mathbf{x}$ and estimating $n = N/V$. Theoretically, we should take the limit as $V \to 0$, but that cannot really work once the volume gets too small: as the edge of the volume shrinks past the location of an individual, the estimated density jumps by a factor $(N-1)/N$ which is not close to 1. Likewise, the definition of a derivative, which in mathematics requires the limit as the separation goes to zero, must be modified. In both cases, we can only require that the limits be well-behaved and converge as the volumes or distances approach some minimum scale. That scale L_{min} must be big enough so that statistical fluctuations are not overwhelming, yet small enough so that variability on oceanographically relevant scales can still be resolved.

Mathematically, we have our estimate

$$n = \langle n \rangle + \frac{N - \langle N \rangle}{V} \equiv \langle n \rangle + n'$$

and we can use the relative sizes of the fluctuations (in an r.m.s. sense) compared to the mean as a measure of our estimation accuracy. Assuming a Poisson distribution[†] for N gives

$$E = \frac{\sqrt{\langle n'^2 \rangle}}{\langle n \rangle} = \frac{\sqrt{\langle (N - \langle N \rangle)^2 \rangle}}{\langle N \rangle} = \frac{\sqrt{\langle N^2 \rangle - \langle N \rangle^2}}{\langle N \rangle} = \frac{\sqrt{\langle N \rangle}}{\langle N \rangle} = (\langle n \rangle V)^{-1/2}$$

Thus our error grows as $V^{-1/2}$ and becomes intolerable for very small volumes.

Conversely, making the volumes too large also becomes problematical: if there is real spatial variation in the density, we will wipe it out by averaging over too large a volume. If there is curvature, the estimated values of n will also be affected. To see this, suppose that the expected value of n satisfies

$$\langle n \rangle = \overline{n}(\mathbf{x}) = n_0 + a_i x_i + \frac{1}{2} b_{ij} x_i x_j$$

Our estimated density would be

$$\hat{n} = \frac{1}{V} \iiint \langle n \rangle$$

For a cube with side $V^{1/3}$, we get

$$\hat{n} = n_0 + \frac{1}{24} b_{ii} V^{2/3} = \overline{n}(0) + \frac{1}{24} \nabla^2 \overline{n}(0) V^{2/3}$$

[†] Given an expected number $\langle N \rangle$ of items, the probability of finding N is $\mathcal{P}(N) = \langle N \rangle^N \exp(-\langle N \rangle)/N!$. From this formula, the expected number is just $\langle N \rangle$ and the expected value of N^2 is $\langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle$

so that the error grows as $V^{2/3}/L^2$ where L is the characteristic length scale of the motion (i.e., $|\nabla^2 \overline{n}| \sim |\overline{n}|/L^2$).

To get errors less than 1%, we need

$$\frac{1}{\overline{n}^{1/2}} < 10^{-2} V^{1/2} \quad \text{and} \quad \frac{V^{2/3}}{24 L^2} < 10^{-2}$$

$$\Rightarrow \quad \frac{10^4}{\overline{n}} < V < 24^{3/2} 10^{-3} L^3$$

$$\Rightarrow \quad \overline{n} L^3 >> 10^5$$

so that there are at least 10^5 organisms within volumes which are close enough to have negligible environmental variations. When these conditions are satisfied, we can extrapolate the measurements back to $V = 0$ and model the set of individuals as a continuous field $n(\mathbf{x}, t)$.

If we apply this argument to fluid properties, where we might typically have $xx \times 10^{22}$ water molecules or $xx \times 10^{20}$ "salt" molecules per cubic centimeter, we can expect the continuum model to be appropriate for length scales above 10^{-5} cm

1.2.1 — Probability Densities

Alternatively, we can view $b_{euph}(\mathbf{x}, t)\ d^3\mathbf{x}$ as representing the probable mass in volume $d^3\mathbf{x}$,

$$b_{euph}(\mathbf{x}, t) = m_i \mathcal{P}_i(\mathbf{x}, t)$$

where $\mathcal{P}_i(\mathbf{x}, t)\ d^3\mathbf{x}$ is the probability that the i^{th} individual is in the volume $d^3\mathbf{x}$ surrounding point $\mathbf{x}$. Now there is no difficulty about the field being continuous; instead, the problem shifts to (1) deciding whether modelling only the mean of the probability distribution is sufficient or whether other moments should be predicted, (2) calculating the probability of interactions between various groups (e.g. predation), and (3) deciding whether observed biomass in a given volume is consistent with the modelled values b_{euph} or not. Although the last issue can be assessed better if we have estimated more than the first moment of the probability distribution, such calculations are rarely part of a model (and, as we shall see, raise other difficulties in terms of closing a moment hierarchy).

1.3 — Formulation of reaction-advection-diffusion equations

Given the continuum approximation, we can generate models for properties such as the biomass concentration within a spatially and temporally fixed volume $dV = dx\, dy\, dz$ with its center at $\mathbf{x}$. A characteristic of the material, b, measured per unit volume, will change by (1) internal sources/ sinks or (2) transport in and out of the volume across the faces.

1.3.1 — Reaction terms

The physical problem involves finding the velocities $\mathbf{u}$ and mixing rates κ (viewed as vector or scalar† fields – functions of $\mathbf{x}$ and t), given the forcing; however, as part of the process, we may have to look at other properties such as temperature, salinity, density ... The biological dynamics, which we express as

$$\frac{\partial}{\partial t} b_i = \mathcal{B}_i(b_1, b_2, b_3, \ldots | \mathbf{x}, t)$$

poses much trickier questions:

- What variables (sets of b's) shall we choose for the various fields in the model? Does b_1 measure the amount of biomass in some functional group (such as photosynthesizing phytoplankton), or something quite different, such as the number of copepods in a weight range of xx to xx mg.
- What kinds of interactions $\mathcal{B}_i(b_1, b_2, b_3, \ldots | \mathbf{x}, t)$ increase or reduce the concentration of b_i (presumably at the expense of some of the other fields). Predation, for example, removes a certain amount of biomass from the prey field (e.g., b_1) and increases the biomass in the predator field (b_2). If the probability that an individual predator finds a prey item is proportional to the prey density, then the rate at which biomass is transferred is proportional to the product of the densities $b_1 b_2$. Other processes happen within a single group – reproduction or death – and thereby appear as terms linear in the density. (In actuality, such assumptions mean that the required resources, such as nutrients, are always adequate and available. In the case of death, a linear term could represent a natural lifetime or it could model predators which forage through some volume of water in a given time period and remove some fraction of the organisms present.)

An example

For a simplified example of a biological model, let us consider an NPZ system: nutrients ($N \equiv b_1$) are taken up by phytoplankton ($P \equiv b_2$), which are grazed upon by zooplankton ($Z \equiv b_3$). The rate at which the net biomass of phytoplankton is reduced depends upon the product of the concentrations of the phytoplankton and the zooplankton – essentially assuming that the grazing increases proportional to the number of encounters between predator and prey during a short time interval. A fraction a of the grazed material is assimilated, while the rest, along with any dead zooplankton, is recycled immediately back into dissolved nutrients. In later chapters, we shall discuss the many oversimplifications of this model and ways of improving and expanding it; for now, the NPZ system serves to introduce the basic approach.

We must define precisely what we mean by a variable such as P. We could take it to be the net biomass per unit volume of water in all the primary producers – the photosynthesizing organisms. Or it could represent the density of some particular species, sets of species, or range of sizes. Or we could try to frame the model in terms of chlorophyll fluorescence. However, we choose an approach based on a "currency" – a property, such as the

† Strictly speaking the mixing may be described as a tensor.

number of moles of nitrogen, which is conserved during interactions. When, for example, a zooplankter feeds on a phytoplankton cell, some of the prey's nitrogen is incorporated into the zooplankter's biomass and some may become detritus or be excreted. In theory, though, we can account for all of it. To relate such variables to measurements, we need either to measure the amount of nitrogen directly or use formulae which express the ratio of properties such as wet weight or carbon content to the number of moles of nitrogen. Such conversions can be difficult when the variable represents a changing assemblage of different species; however, as noted by Redfield (19xx), some ratios are relatively constant across a wide range of organisms.

Based on nitrogen as a currency, then, we can make the biological dynamics more specific, using a combination of linear and quadratic rules: the phytoplankton grow by taking up nutrient and are grazed by the zooplankton

$$\frac{\partial}{\partial t}P = \mu PN - gPZ - d_P P \tag{1.1}$$

The zooplankton assimilate a fraction of the grazed material and lose nitrogen by respiration and grazing by predators

$$\frac{\partial}{\partial t}Z = agPZ - d_Z Z \tag{1.2}$$

The nutrient is lost to phytoplankton and regenerated from the unassimilated grazed material and the decay of zooplankton

$$\frac{\partial}{\partial t}N = -\mu PN + (1-a)gPZ + d_P P + d_Z Z \tag{1.3}$$

In chapter xx, we shall examine the construction of biological models in much more detail and discuss approaches for remedying the all too numerous flaws in (1.1–1.3) which we will call the Quadratic NPZ model. Some of these problems include:

- The term "phytoplankton" stands for a wide variety of different species. If the different varieties always appeared in the same ratio, one might begin to believe that a single growth rate μ for the assemblage as well as a simple grazing rule could indeed be used. But different phytoplankton species are dominant at different times and/or in different places; how much of this diversity must be included before we can understand the system better?
- Nutrients, too, come in many flavors. Dissolved nitrogen appears as both nitrate, nitrite, and ammonia, with different organisms having different abilities to use these variants. And phytoplankton do not live by nitrogen alone; over the past xx years, the work of Martin(19xx) and others has demonstrated that growth is often limited by the scarcity of iron rather than of nitrogen.
- Zooplankton not only come in many different species, they may also have a complicated life cycle, with changes in form, behavior, foraging ability, and size. Passing through these stages may take a year; some copepods, for example, have a dormant stage for over-wintering. Can we learn much by using a single variable Z and glossing over all the complexity, or do we need to include at least some of this kind of detail.

- Can we close the system with the zooplankton, representing their losses by predation and death as $d_Z Z$, or do we need to include higher trophic levels? For that matter, what about bacteria, protozoa, ...?

Many of these issues depend upon the question being asked; the choices made will depend on the degree to which one is trying to make a model for a small-scale ecosystem perhaps with interests in only a limited component (e.g., copepods in Massachusetts Bay) or a global scale model to look at overall distributions.

In addition, there are mathematical issues to consider:

- What are appropriate functional forms $\mathcal{B}_i$? How do the characteristics of the dynamics change as we alter them?
- What are appropriate values for the coefficients such as g? Can these be estimated from laboratory experiments or by fitting the model to data?
- As the model becomes more complex, we inevitably introduce more and more poorly known constants and functional forms. At what point are we putting in so much that the model becomes statistically meaningless?
- How do we verify a model? While we may be able to show that a model utterly fails, most comparisons with data (and the data, of course, is still extremely sparse) tend to be of the "this looks like that" variety. Even more rigorous statistical comparisons cannot validate a model, since more than one dynamical system may produce similar enough behavior (see xx).

The Quadratic NPZ Model

With these questions deferred, let us examine the behavior of the Quadratic NPZ model. The system has five parameters ($\mu,\ g,\ d_P,\ a,\ d_Z$) as well as one which is less obvious, stemming from the closed nitrogen cycle. If we sum the three equations, we find

$$\frac{\partial}{\partial t}(N + P + Z) = 0$$

so that the initial value of $N_T = N(0) + P(0) + Z(0)$ is unchanged during the evolution. We can incorporate that information directly and reduce the number of equations to two:

$$\begin{aligned} \frac{\partial}{\partial t}P &= P\big[\mu(N_T - P - Z) - gZ - d_p\big] \\ \frac{\partial}{\partial t}Z &= Z\big[agP - d_Z\big] \end{aligned} \qquad (\mathrm{Q-NPZ})$$

Steady states:

We begin the analysis of such a system by looking for the equilibrium or steady-state solutions. Setting the right-hand sides of (Q-NPZ) to zero gives three such states

1) $P = Z = 0,\ N = N_T$
2) $Z = 0,\ N = d_P/\mu,\ P = N_T - d_P/\mu \quad$ for $N_T > d_P/\mu$
3) $P = d_Z/ag,\ Z = (\mu N_T - \mu d_Z/ag - d_P)/(\mu + g),\ N = N_T - P_Z \quad$ for $N_T > d_Z/ag + d_P/\mu$

Figure 1.2 summarizes the behavior of this system of equations as one parameter (μ) is varied: we have steady states with only N for small uptake rates, then states with both N and P, and finally states with non-trivial values for all three variables when the uptake rate becomes high enough. Time evolution pictures also show that the approach to steady state occurs via a decaying oscillation for small μ values. Since we expect nutrient uptake rates to decrease as the light level decreases (although it may also decrease when the light is too strong), we can think of these figures as showing the effects of depth changes. Indeed if $\mu = \mu_0 \exp(z/h_\ell)$ where the e-folding scale for the light is h_ℓ (for which we'll use 17 m), we find the steady state shown in figure 1.3.

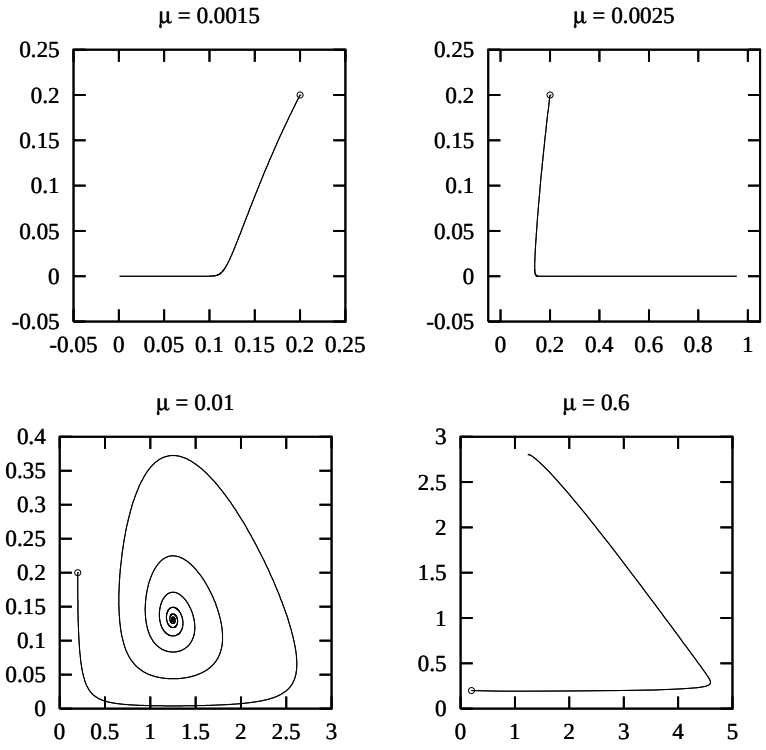

Figure 1.2: Behavior for various μ values. The starting conditions are $P = Z = 0.2$.

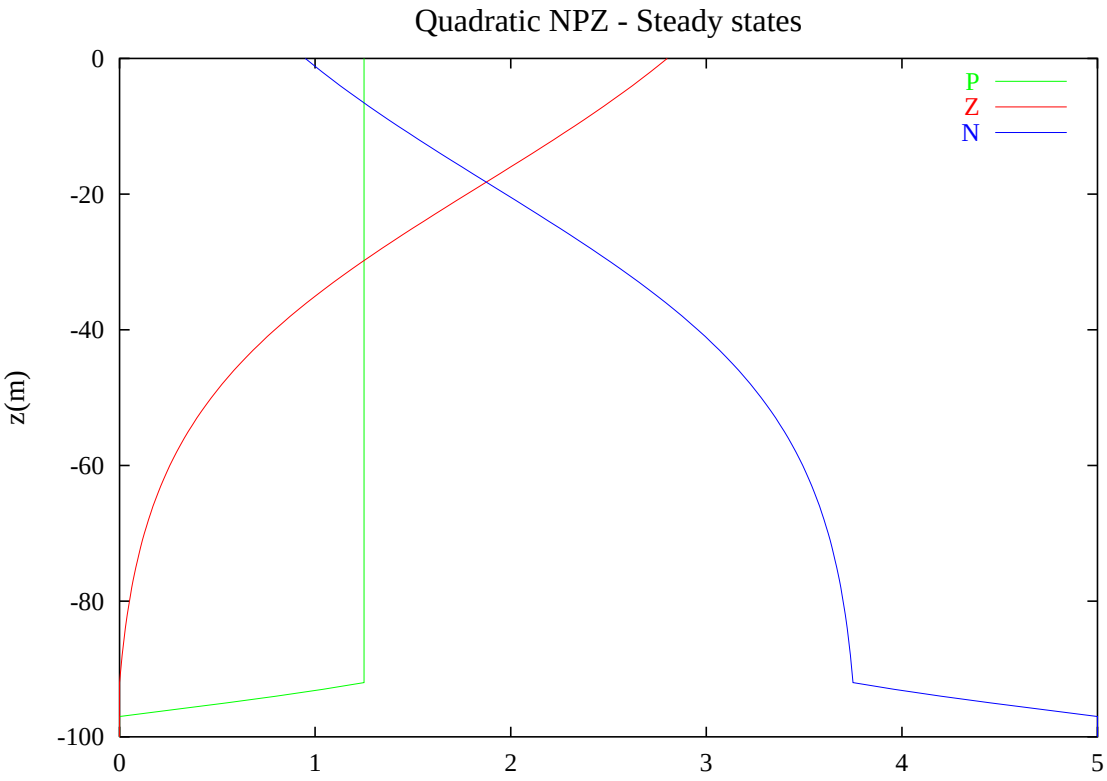

Figure 1.3: Equilibrium values versus μ and depth

Stability

Once we know the steady states, we must also consider their **stability**: what happens if we perturb the fields slightly away from the equilibrium values? Movement and mixing in the ocean provide such perturbations: if we consider transporting some water and organisms vertically in the situation shown in figure 1.3, the values will no longer be consistent with the local equilibrium conditions. By studying the stability, we can gain insight into the ways the system will adjust under physical forcing. (We do, however, have to recognize that the physically-induced perturbations may not always be small; later, we shall look at the conditions under which this kind of approximation may or may not be valid).

The steady states are defined by

$$\mathcal{B}_i(\overline{b}_1(\mathbf{x}), \overline{b}_2(\mathbf{x}), ..., \overline{b}_N(\mathbf{x})|\mathbf{x}) = 0$$

(under the assumption that the reaction terms have no explicit time-dependence (c.f. section xx). If we perturb each field

$$b_i = \overline{b}_i(\mathbf{x}) + b_i'(\mathbf{x}, t)$$

we have

$$\frac{\partial}{\partial t}b'_i = \mathcal{B}_i(\overline{b}_1 + b'_1, \overline{b}_2 + b'_2, ..., \overline{b}_N + b'_n) - \mathcal{B}_i(\overline{b}_1, \overline{b}_2, ..., \overline{b}_N)$$

(letting the $\mathbf{x}$ dependence be implicit). If the perturbations are small, we can Taylor-expand the right hand side to find

$$\frac{\partial}{\partial t}b'_i \simeq b'_1 \frac{\partial}{\partial \overline{b}_1}\mathcal{B}_i(\overline{b}_1, \overline{b}_2, ..., \overline{b}_N) + b'_2 \frac{\partial}{\partial \overline{b}_2}\mathcal{B}_i(\overline{b}_1, \overline{b}_2, ..., \overline{b}_N)... + b'_N \frac{\partial}{\partial \overline{b}_N}\mathcal{B}_i(\overline{b}_1, \overline{b}_2, ..., \overline{b}_N)$$

so that the perturbations satisfy the matrix equation

$$\frac{\partial}{\partial t}b'_i = \mathcal{B}_{ij}b'_j \quad , \quad \mathcal{B}_{ij} \equiv \frac{\partial \mathcal{B}_i}{\partial b_j}\Big|_{\mathbf{b}=\overline{\mathbf{b}}} \tag{1.4}$$

We can write

$$b'_i(t) = \exp[t\mathcal{B}_{ij}]b'_j(0)$$

where the matrix exponential can be defined in terms of the eigenvalues and eigenvectors of $\mathcal{B}_{ij}$:

$$\exp(t\mathcal{B}_{ij}) = v_{ik}^{-1} e^{\sigma_k t} v_{kj}$$

Here the k^{th} column of v_{ik} is the eigenvector corresponding to eigenvalue σ_k

$$\mathcal{B}_{ij}v_{jk} = v_{ik}\sigma_k$$

When the real part of every eigenvalue is negative, all perturbations will eventually decay and we call the equilibrium stable; if the real part of one or more is positive, that mode will grow to large amplitude (when the Taylor approximation breaks down) and we call the equilibrium unstable. Purely imaginary eigenvalues correspond to neutral modes which simply oscillate; the linearized equations are not entirely adequate. As we shall discuss later, the transient behavior even of a stable system can show considerable temporary growth, so that an asymptotic stability result may be misleading, but it is at least a start.

For the Q-NPZ model, the matrix $\mathcal{B}_{ij}$ is

$$\mathcal{B}_{ij} = \begin{pmatrix} \mu(N_T - \overline{P} - \overline{Z}) - g\overline{Z} - d_P - \mu\overline{P} & -(\mu+g)\overline{P} \\ ag\overline{Z} & ag\overline{P} - d_Z \end{pmatrix}$$

which simplifies for the three equilibria to

$$\overline{P} = \overline{Z} = 0 \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} \mu N_T - d_P & 0 \\ 0 & -d_Z \end{pmatrix}$$

$$\overline{Z} = 0,\ \overline{P} = N_T - d_P/\mu \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} -\mu\overline{P} & -(\mu+g)\overline{P} \\ 0 & ag\overline{P} - d_Z \end{pmatrix}$$

$$\overline{P} = d_Z/ag,\ \overline{Z} = (\mu N_T - \mu d_Z/ag - d_P)/(\mu+g) \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} -\mu\overline{P} & -(\mu+g)\overline{P} \\ ag\overline{Z} & 0 \end{pmatrix}$$

For a 2×2 real matrix, at least one of the eigenvalues will have a positive real part if the trace (Tr, the sum of the diagonal elements) is positive or the determinant (Det, the product of the diagonal elements minus the product of the off-diagonal elements) is negative. Thus the $\overline{P} = \overline{Z} = 0$ state becomes unstable when $\mu N_T > d_P$ – phytoplankton have enough nutrient to grow. The $\overline{P} \neq 0$, $\overline{Z} = 0$ state will become unstable when $ag\overline{P} > d_Z$; at this point, the phytoplankton have sufficient biomass to support zooplankton growth. In both cases, the determinant switches from positive to negative before the trace becomes positive. Finally, the state with both $\overline{P}$ and $\overline{Z}$ non-zero will always have negative Tr and positive Det, so that all perturbations will decay.

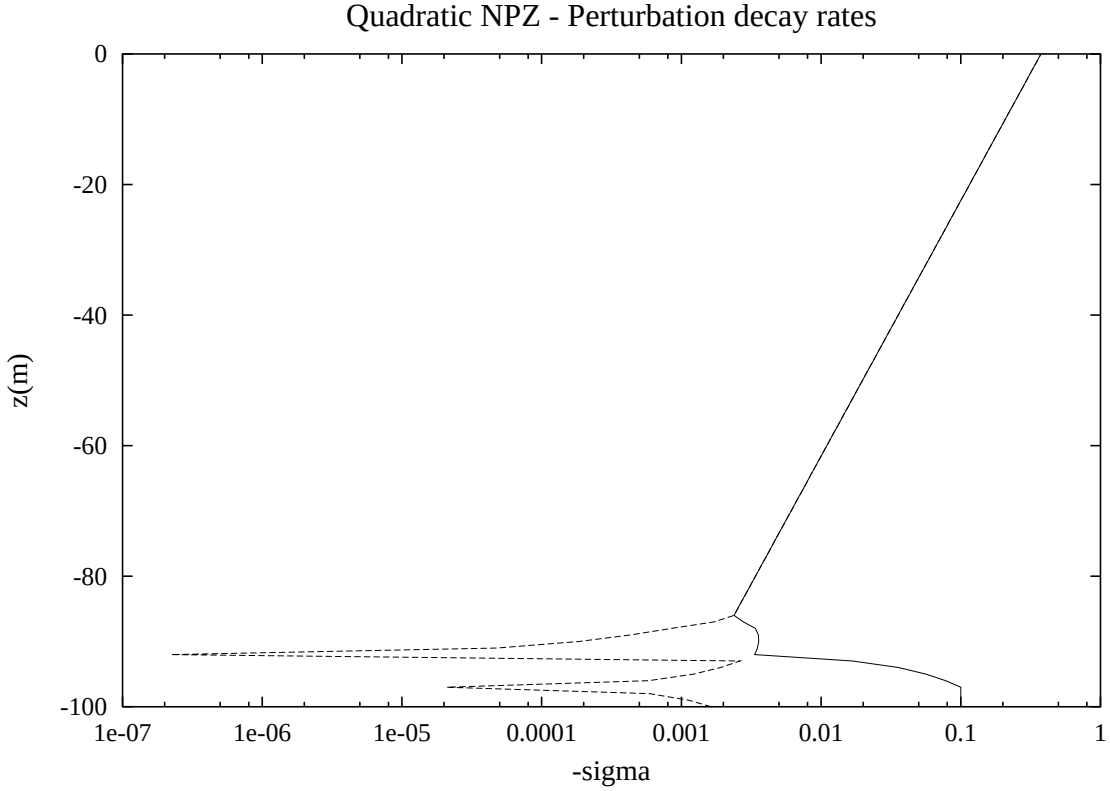

Figure 1.4: Decay rates for perturbations ($-\sigma$, units d^{-1})

1.3.2 — Fluxes: Advection

Now let us consider changes in a spatially fixed volume associated with transport by the fluid motion (called "advection"). To calculate these, we consider the flux of b across a surface with outward normal $\hat{\mathbf{n}}$. The volume of fluid crossing through the surface is given by the area times the height measured normal to the surface, and the latter is just the normal component of the velocity multiplied by the time interval (figure 1.5).

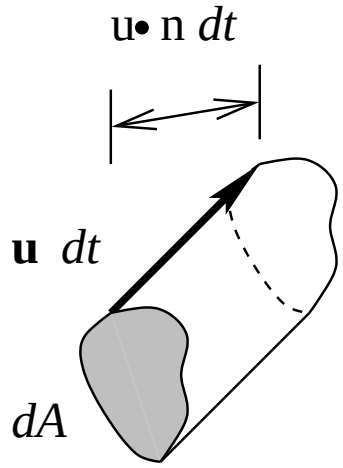

Figure 1.5: Flux through an area

Thus the volume of fluid passing out of the monitored area is

$$\mathbf{u} \cdot \hat{\mathbf{n}}\, dA\, dt$$

and the flux (amount passing through a unit area in a unit time) of b is

$$F = b(\mathbf{x})\mathbf{u} \cdot \hat{\mathbf{n}}$$

If we now look at the rate of change of the integrated amount of b, we have

$$\frac{d}{dt}\int_V b\; d^3\mathbf{x} = -\int_{\partial V} b\mathbf{u} \cdot \hat{\mathbf{n}}\; d^2\mathbf{x} + \int_V \mathcal{B}\; d^3\mathbf{x}$$

where $\mathcal{B}$ represents the rate of production/ destruction per unit water volume. We use the divergence theorem and take the limit as $V \to 0$ to find

$$\frac{\partial}{\partial t} b = -\nabla \cdot \mathbf{u} b + \mathcal{B} \tag{1.5}$$

We shall use this with appropriate source/sink functions to derive the fluid equations.

Note: Einstein summation notation – we shall often write equations like the above in terms of the Cartesian indices and use the convention that an index appearing multiple times in one term and not in others is summed over:

$$\frac{\partial}{\partial t} b = -\frac{\partial}{\partial x_i}(u_i b) + \mathcal{B} \quad \left[meaning\ \frac{\partial}{\partial t} b = -\sum_i \frac{\partial}{\partial x_i}(u_i b) + \mathcal{B}\right]$$

or

$$\frac{\partial}{\partial t} b_i = -\frac{\partial}{\partial x_j}(u_j b_i) + \mathcal{B}_i \quad \left[meaning\ \frac{\partial}{\partial t} b_i = -\sum_j \frac{\partial}{\partial x_j}(u_j b_i) + \mathcal{B}_i\right]$$

Conservation of mass

For mass, we have $b = \rho$ the water density (mass per unit volume) and no sources or sinks $\mathcal{B} = 0$. Therefore

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot (\rho \mathbf{u}) \tag{1.6}$$

Scalar properties

Fluid momentum (governed by Newton's equations) requires some extra caution, since momentum, like velocity, is a **vector** property related to the vector positions; i.e., the value of the x-component of the momentum depends on how we choose to orient our coordinate system and how that orientation changes with (for example) changes in latitude or longitude. In contrast, the biological and chemical (as well as temperature and salinity) are **scalars** and do not depend on whether we decide x means east or whether it means distance perpendicular to the shore.

For scalar properties, such as salinity S which is expressed as mass of salt per mass of water, we can combine the conservation equation with the conservation of mass to find

$$\frac{\partial}{\partial t}\rho S = -\nabla(\mathbf{u}\rho S)$$

or

$$\frac{\partial}{\partial t}S + \mathbf{u} \cdot \nabla S = 0$$

We define the operator for the "derivative following the fluid" (also known as the material or substantial derivative) by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

We can understand this operator by considering how a scalar changes, looking at two nearby time points for the same small mass or blob of fluid (rather than the same region of space). We then have

$$S(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) = S(\mathbf{x}, t)$$

(figure 1.6).

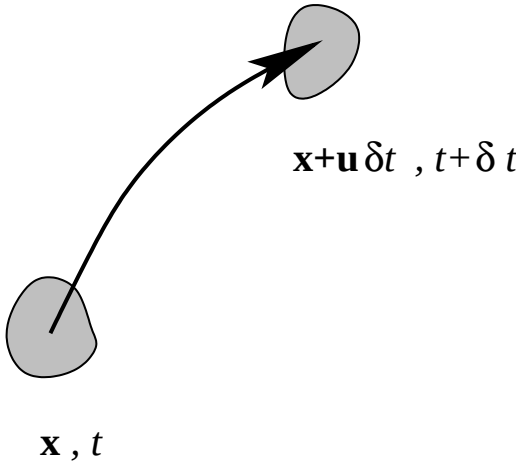

Figure 1.6: Lagrangian view

Taylor-expanding for small δt gives

$$S(\mathbf{x},t) + \left[\frac{\partial}{\partial t}S + u\frac{\partial}{\partial x}S + v\frac{\partial}{\partial y}S + w\frac{\partial}{\partial z}S\right]\delta t = S(\mathbf{x},t)$$

which results in the same equation derived above.

In the case of a property expressed per unit volume, our advection equation can be written

$$\frac{D}{Dt}b = -b\nabla\cdot\mathbf{u} + \mathcal{B} \tag{1.7}$$

Since the fluid velocity divergence is very small, the first term on the right is significant only when the material moves significantly relative to the water.

The relationship between this form and the derivation following a blob of fluid is slightly more complex: we now have

$$b(\mathbf{x}+\mathbf{u}\delta t, t+\delta t)V(\mathbf{x}+\mathbf{u}\delta t, t+dt) = b(\mathbf{x},t)V(\mathbf{x},t) + V(\mathbf{x},t)\mathcal{B}(\mathbf{x},t)\delta t$$

In the limit as $\delta t \to 0$, we find

$$V\frac{D}{Dt}b + b\frac{D}{Dt}V = V\mathcal{B} \quad \Rightarrow \quad \frac{D}{Dt}b = -b\frac{D}{Dt}\ln V + \mathcal{B}$$

For a constant-mass blob of fluid, $\rho V = \text{const.}$, we can see that $\frac{D}{Dt}\ln V = -\frac{D}{Dt}\ln\rho = \nabla\cdot\mathbf{u}$, and we recover equation (1.7). Note that we can also express this in the form

$$\frac{D}{Dt}\frac{b}{\rho} = \frac{1}{\rho}\mathcal{B}$$

1.3.3 — Fluxes: Diffusion

Diffusion is a complicated topic for ocean models, since they rely on a parameterization of the way turbulence at smaller, unresolved scales moves material around. For organisms, their own motion relative to the water can also contribute significantly to the transport; we shall address this issue in more detail below. For now, we simply assume that in a time δt, a mass of fluid $\delta M = \rho\, \delta A\, \delta s$ passes out of the region through an area δA to be replaced by an identical mass from outside. (Any net imbalance in mass would be associated with the fluid velocity $\mathbf{u}$ by definition – the velocity is the weighted average of the momenta of the individual molecules divided by the net mass $\mathbf{u} = \sum m_i \mathbf{u}_i / \sum m_i$. Accordingly, the fluctuations – some molecules going left and some right over and above the mean – have no net mass flux.)

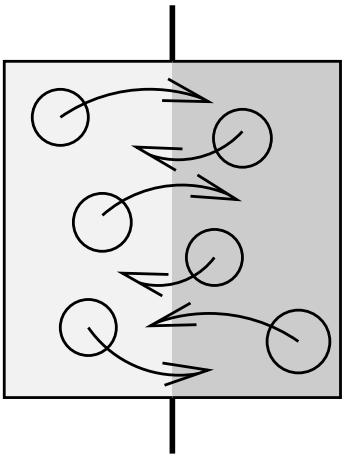

Figure 1.7: Diffusive interchange of material

The flux of b associated with the back-and-forth exchange is

$$[\delta A\, \delta s_{inside}\, b_{inside} - \delta A\, \delta s_{outside}\, b_{outside}]/\delta A\, \delta t = \left[\frac{b_{inside}}{\rho_{inside}} - \frac{b_{outside}}{\rho_{outside}}\right] \frac{\delta M}{\delta A\ \delta t}$$

Taylor-expanding and using the fact that the distance between the center points of the two masses is $\ell \equiv \delta M/\rho\delta A = \delta s$ gives

$$\mathbf{F} = -\frac{\delta M^2}{\rho \delta A^2 \delta t} \nabla \frac{b}{\rho} = -\rho\kappa \nabla \frac{b}{\rho}$$

where the diffusivity is $\kappa = \ell^2/\delta t$. We can interpret ℓ as the mean free path over a decorrelation time δt. If we add this term to the scalar equation, we have the reaction-advection-diffusion system:

$$\frac{\partial}{\partial t} b = -\nabla \cdot [\mathbf{u} b - \rho\kappa \nabla \frac{b}{\rho}] + \mathcal{B}$$

Customarily, the density terms are neglected in the diffusion, since the variations are slight (although the first form makes it clear that at least the ρ^{-1} factor will be required in order that the net amount of b mass $\int d^3\mathbf{x}\rho b$) (e.g., in the case of salinity, the number of "salt" molecules) will be conserved. As we shall see, the distinction between volume and mass in this equation becomes irrelevant when we use the Boussinesq equations or pressure coordinate models. Hereafter, we will deal with the reaction- advection- diffusion equations

$$\frac{\partial}{\partial t}b + \nabla \cdot (\mathbf{u}b) = \nabla \cdot \kappa \nabla b + \mathcal{B} \qquad (RAD)$$

For quantities defined per unit mass, the equations are similar

$$\begin{aligned}\frac{D}{Dt}S &= \frac{1}{\rho}\nabla \cdot [\rho\kappa\nabla S] \\ &\simeq \nabla \cdot \kappa \nabla S\end{aligned}$$

- Diffusive laws are often used to model turbulent transport; however, as discussed in chapter xx, the basic assumption that the mean free path of a blob of fluid is small compared to the scale of variation of properties (i.e, that there is a significant scale separation between the sizes of eddies and the mean) is frequently not appropriate.

Examples

If we take the steady states shown in figure 1.3, and add increasing amounts of diffusion, we find the discontinuities in vertical gradients begin to smooth out. Organisms can now be found at depths where their net growth rate is negative; the downward flux by mixing balances the losses. As the mixing rate increases, the fields become more and more uniform vertically. The final stage will be almost homogeneous, and the values of the b's will satisfy the biological ODE's with all z-dependent coefficients replaced by their vertical average. For a deep system, the average uptake will not be able to support populations, and the P and Z values will tend towards zero everywhere.

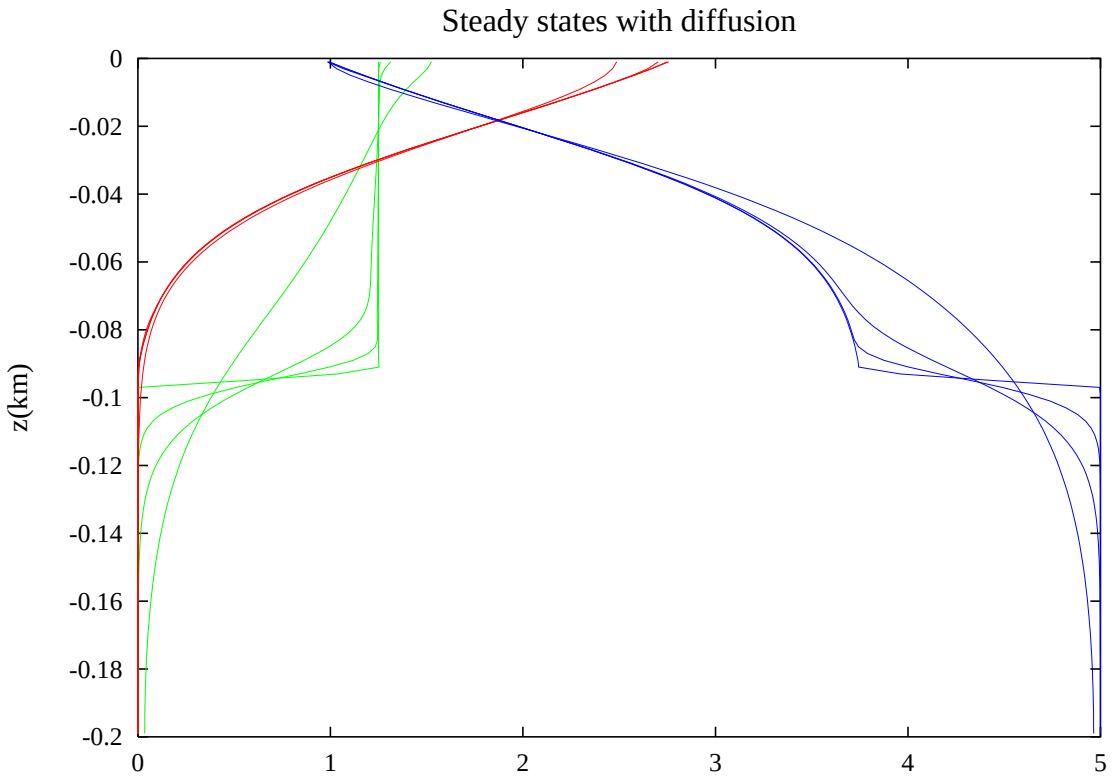

Figure 1.8: κ values of 0 (sharpest), 10^{-7}, 10^{-6}, and 10^{-5} (smoothest and deepest) km^2/d

Advection has somewhat similar, but more complicated, effects. Consider tracking the organisms in a small parcel of fluid over time: if the light level averaged over the trajectory is too small (less than d_P/N_T), they will die out.

To understand trajectories and their relationship to the flow velocity field $\mathbf{u}(\mathbf{x}, t)$ we must examine the two different, but interrelated, ways of analyzing the flow of a gas or liquid: the Lagrangian and the Eulerian descriptions. For the first approach, we follow fluid parcels – think of a small blob of dyed fluid which remains coherent – and calculate the changes in its position $\mathbf{X}(t)$ and other properties. Observationally, we try to do this with various kinds of drifters and subsurface floats. As an analogy, consider a fluid parcel to be like an automobile: the velocity is determined by the speedometer reading and the current road direction. Integrating this in time gives the position of the particular car. Likewise, cars may carry other properties such as the temperature within the passenger compartment which can vary from car to car or with time for an individual vehicle.

In contrast, the Eulerian description gives the properties at fixed positions in space, as though we had moored instruments to measure various characteristics. Thus it represents the fluid properties as fields. In terms of the automobile analogy, the Eulerian velocity would be the quantity measured with radar by the policeman parked at the side of the road. The time history would now be the records of the speeds of the different cars

streaming by the point, not that of any one car's speedometer; however, the two should agree at the instant that car passes the radar. Likewise, radiometer measurements of temperatures in passing cars give a different kind of information from a thermometer in one car: observations by the stationary observer of a decreasing temperature do not mean that the drivers are all turning down their thermostats.

Trajectories are inherently Lagrangian information – they describe where the fluid parcels move, much like ordinary Newtonian dynamics which follows masses as they interact with each other and with external forces. The Lagrangian description can be thought of as a map from a set (continuum) of initial positions $\mathbf{X}_0$ to the positions at time t, $\mathbf{X}(t|\mathbf{X}_0)$.

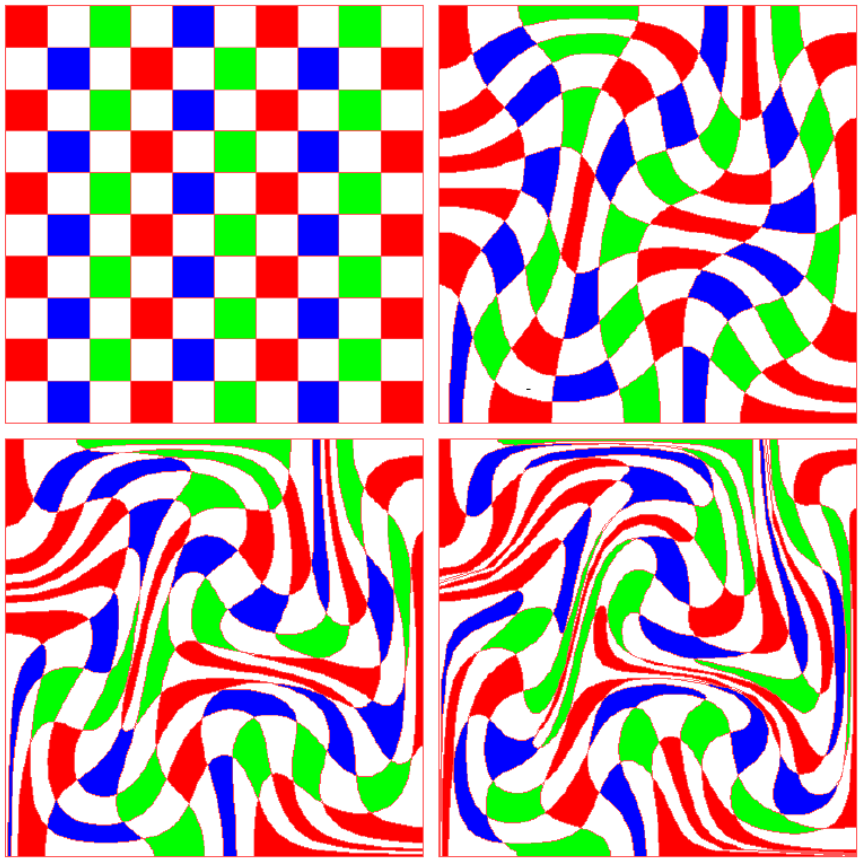

Figure 1.9: Map showing motion and distortion of fluid in a two-dimensional flow

If $\mathbf{X}(t|\mathbf{X}_0)$ represents the position of a small fluid parcel started at position $\mathbf{X}_0$ at $t = 0$, then the velocity is given by $d\mathbf{X}/dt$; this must match the velocity $\mathbf{u}$ at the particle's current location. Thus we have the equation for the trajectory

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t) \tag{1.8}$$

given the Eulerian velocity $\mathbf{u}(\mathbf{x}, t)$. Equation (1.9) is the fundamental link between the Eulerian and Lagrangian descriptions of fluid motion: it says that the rate of change of position for a float passing by a current meter should be the same as the velocity measured by that meter at the time the two instruments meet (in reality, of course, we do not want them to crash into each other, and we expect instrument errors and averaging will make the velocity match imprecise).

For non-divergent, non-diffusive flow, we can readily examine the biological equations in a Lagrangian framework (when they just become one ODE for each fluid parcel):

$$\frac{d}{dt} b_i^L(t) = \mathcal{B}_i(b_1^L, b_2^L, b_3^L, \ldots | \mathbf{X}(t), t) \tag{1.9}$$

with b^L related to our previous density by $b^L(t) = b(\mathbf{X}(t), t)$. Indeed, this equivalence leads naturally back to the substantial derivative:

$$\frac{d}{dt} b_i^L(t) = \frac{\partial}{\partial t} b(\mathbf{X}, t) + \frac{dX_i}{dt} \frac{\partial}{\partial X_i} b(\mathbf{X}, t) = \mathcal{B}(b_1, b_2, b_3 \ldots | \mathbf{X}, t)$$

or

$$\frac{\partial}{\partial t} b + \mathbf{u} \cdot \nabla b = \mathcal{B}$$

as before.

As a beginning step to understanding how pure advection affects the biological dynamics, we can take a simplified analytic form for the velocity fields, calculate the trajectories, and integrate equation (1.10) along them. As we shall see when we examine the physics in more detail, upwelling systems tend to have a shallow region of off-shore flow, with onshore flow in the deeper water, and upward motion near the coast (and the bottom if it is sloped). We examine

$$u = -\frac{p_0}{H}(1 - e^{-X/L_\tau})(1 - e^{Z/h_{ek}}) + \frac{p_0}{h_{ek}}(1 - e^{-X/L_\tau}) e^{Z/h_{ek}} \frac{Z + H}{H}$$

$$w = \frac{p_0}{L_\tau} e^{-X/L_\tau}(1 - e^{Z/h_{ek}}) \frac{Z + H}{H} - p_0(1 - e^{-X/L_\tau})(1 - e^{Z/h_{ek}}) \frac{Z}{H^2} \frac{H_\infty}{L \cosh^2(X/L)}$$

where the depth is given by $H = H_\infty \tanh(x/L)$. This flow has been derived from a streamfunction (section x,x) and is non-divergent ($\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$). We also ensure that there is no normal flow into the bottom or the surface. Trajectories are shown in figure 1.10.

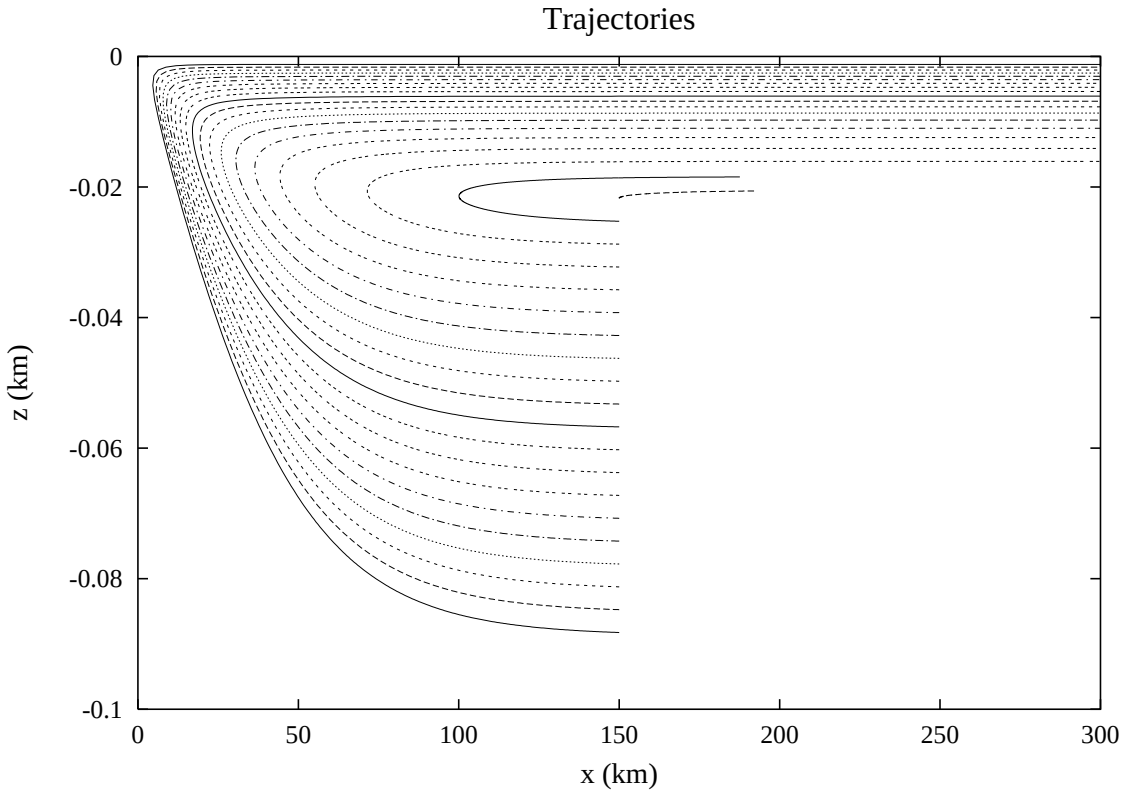

Figure 1.10: Trajectories in upwelling flow.

We now integrate the Q-NPZ model along the trajectories starting with the equilibrium state for the initial depth (figure 1.3). The trajectories starting in the deep water bring fluid with high nutrients and relatively few grazers up into the surface where the light is more intense. The phytoplankton grow rapidly; somewhat later, the zooplankton biomass increases as they graze the phytoplankton back down. As the trajectories head back out to sea, P and Z settle to a new equilibrium corresponding to the current light level (figures 1.11a-d). Spatially, we see a peak in the phytoplankton in the upper 40 meters along the slope at the coast; the zooplankton have values which are somewhat lower in this region.

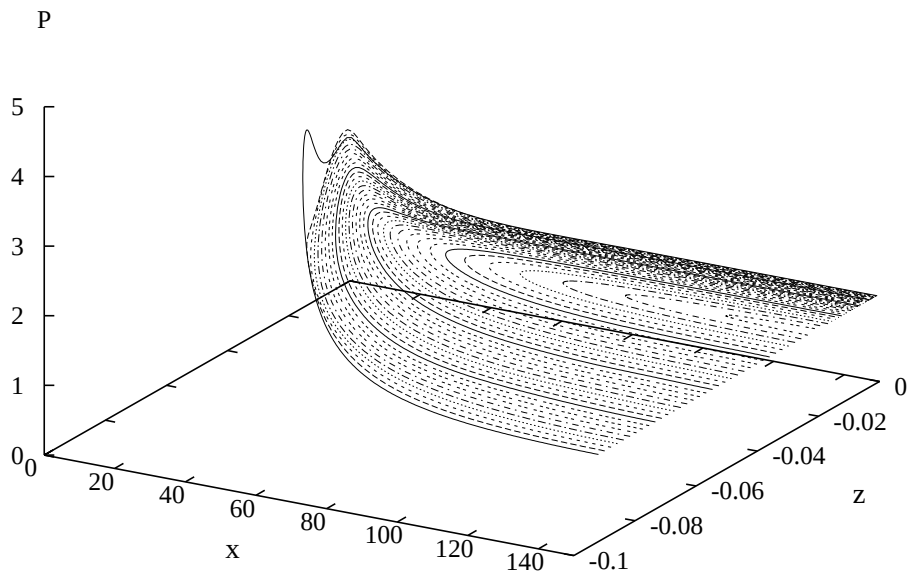

Figure 1.11a: Phytoplankton biomass along trajectories.

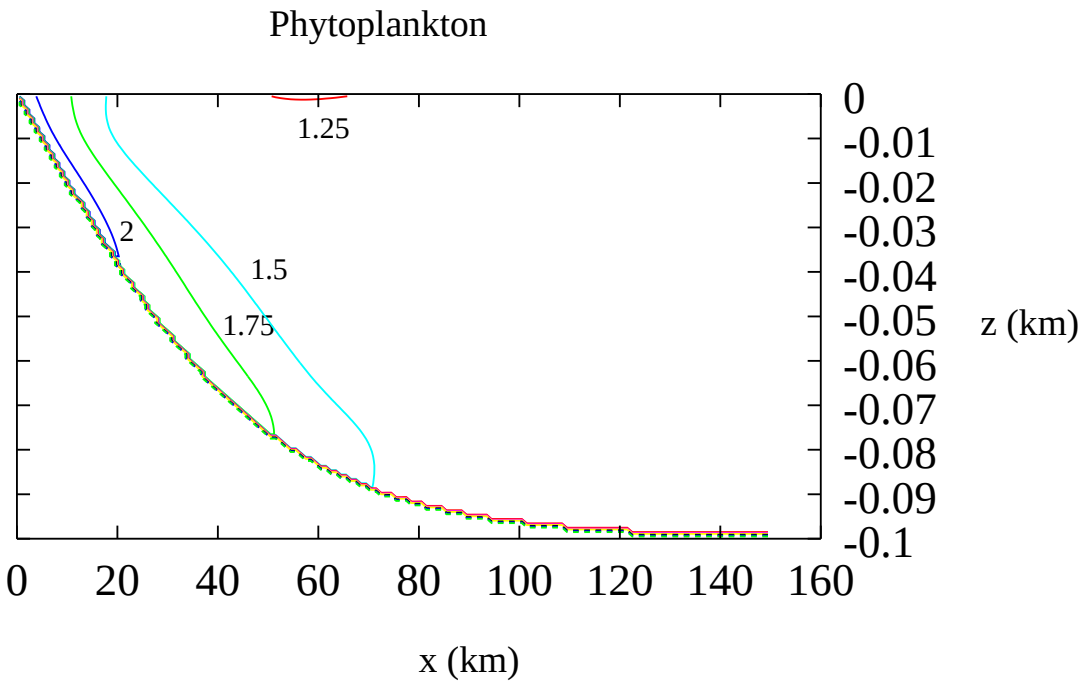

Figure 1.11b: Gridded phytoplankton densities

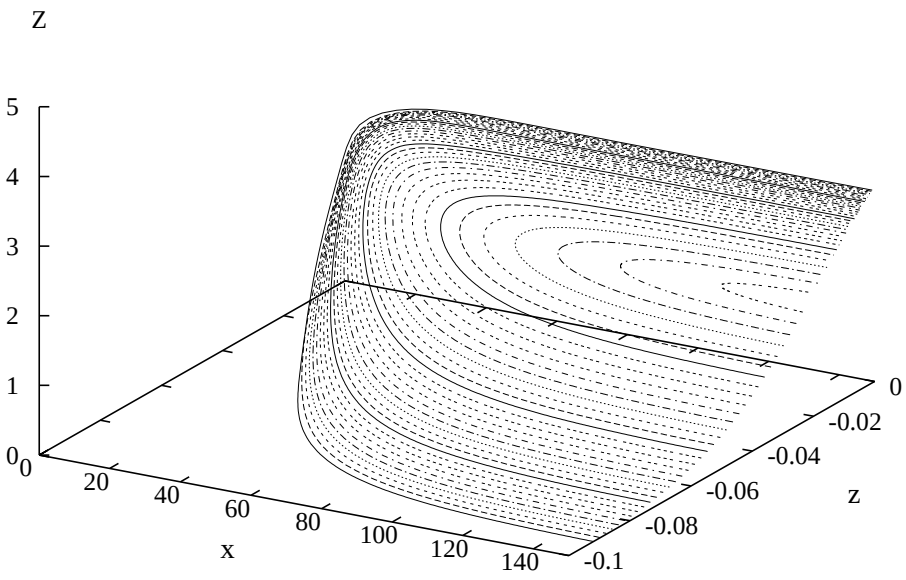

Figure 1.11c: Zooplankton biomass along trajectories.

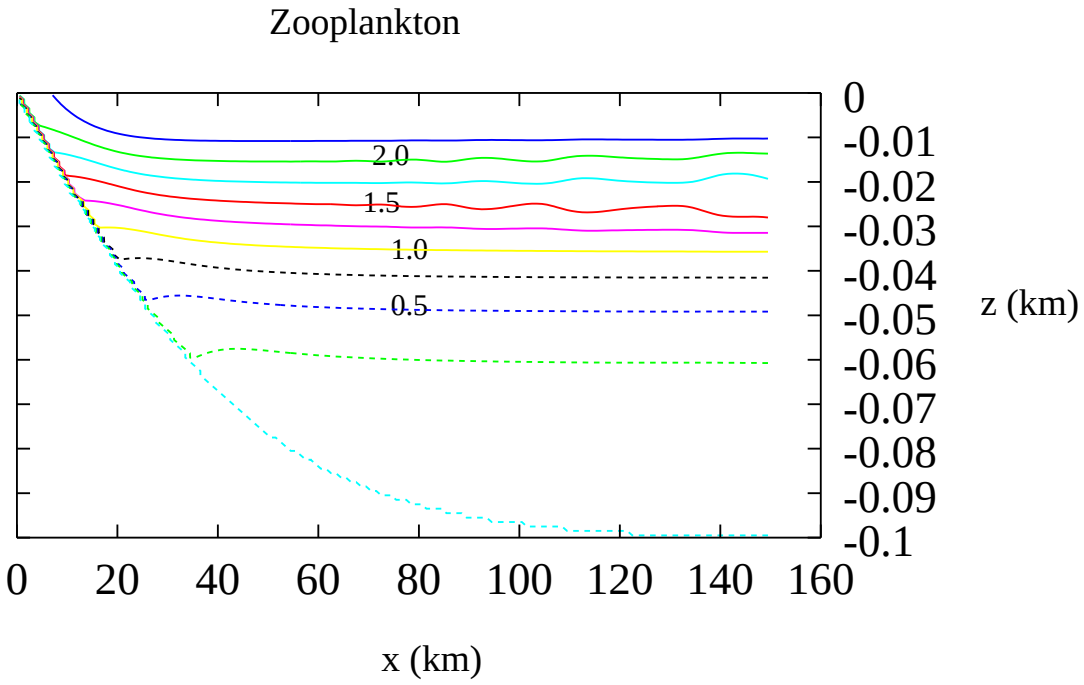

Figure 1.11d: Gridded zooplankton densities

Analysis from linearized biology

As a first attempt at understanding the role of upwelling in an Eulerian framework, we can assume that the physical effects are weak, so that they essentially provide a forcing for perturbations upon a vertically-dependent biological equilibrium. Let us suppose that the basic state includes the vertical diffusion, then the perturbation equations become

$$\frac{\partial}{\partial t} b_i' = Bijb_j' + \kappa \nabla^2 b_i' - \mathbf{u} \cdot \nabla \overline{b}_i$$

For the Q-NPZ model, when all three components exist, the vertical gradients of $\overline{P}$ are zero so that the steady response will satisfy

$$\begin{pmatrix} -\mu\overline{P} & -(\mu+g)\overline{P} \\ ag\overline{Z} & 0 \end{pmatrix} \begin{pmatrix} P' \\ Z' \end{pmatrix} + \kappa\nabla^2 \begin{pmatrix} P' \\ Z' \end{pmatrix} = \begin{pmatrix} 0 \\ w\frac{\partial \overline{Z}}{\partial z} \end{pmatrix}$$

For weak diffusive effects, we see

$$P' = w \frac{1}{ag\overline{Z}} \frac{\partial \overline{Z}}{\partial z} \quad , \qquad Z' = -w \frac{\mu}{\mu+g} \frac{1}{ag\overline{Z}} \frac{\partial \overline{Z}}{\partial z}$$

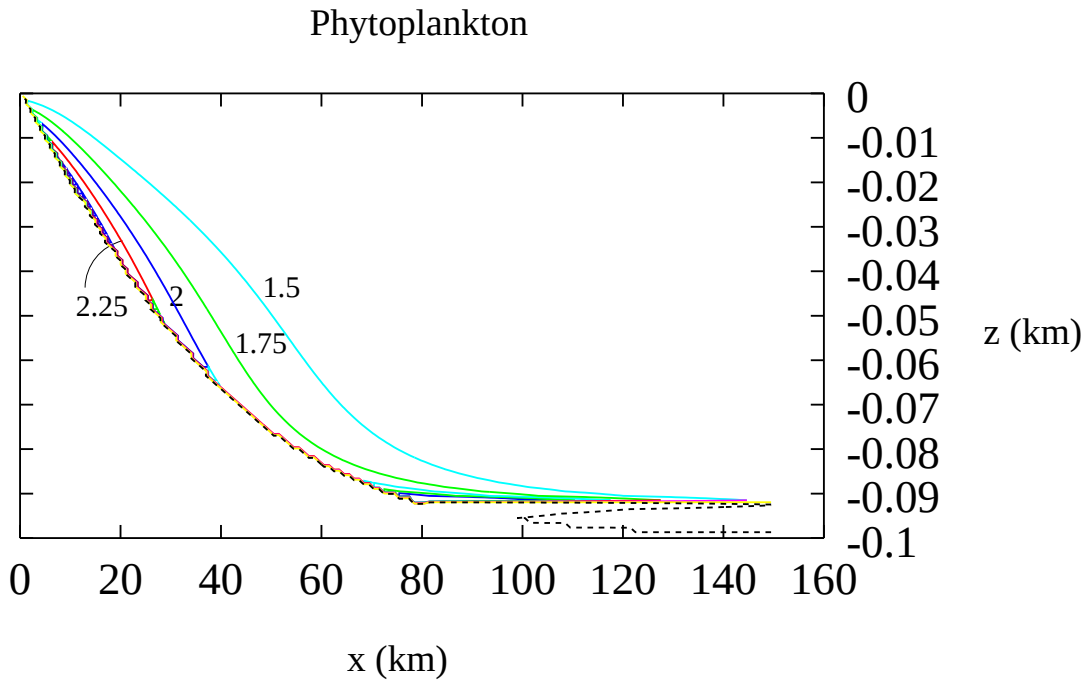

Figure 1.12a: Gridded phytoplankton densities, linearized approx.

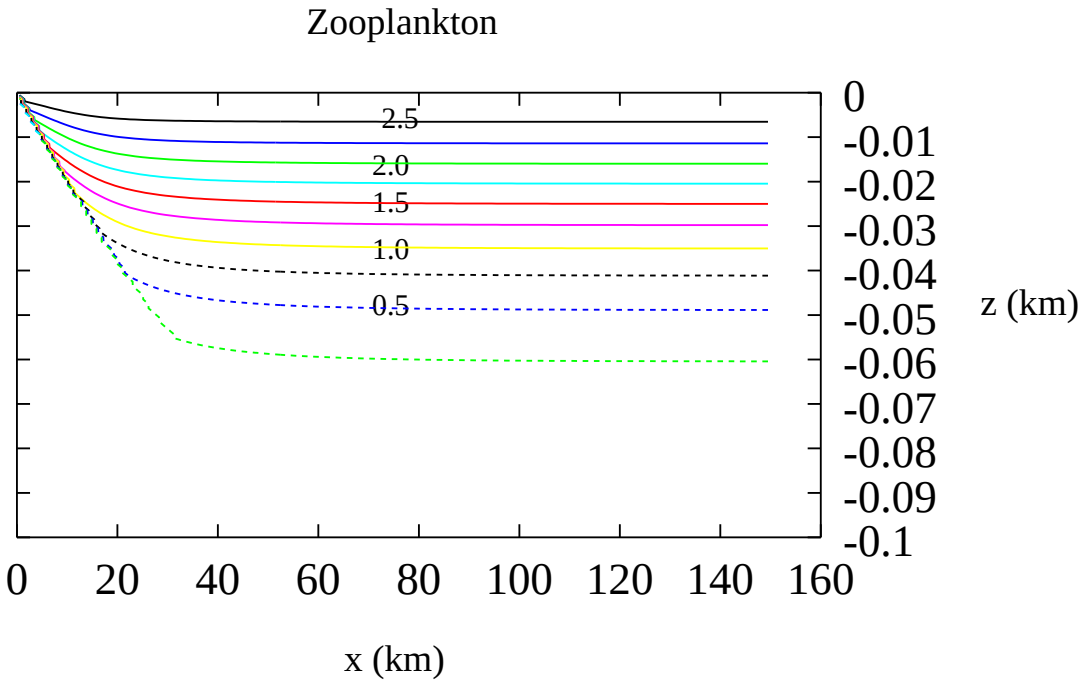

Figure 1.12b: Gridded zooplankton densities, linearized approx.

1.4 — Formulation of the physics

We have already derived the equation for the mass balance in a spatial region: time changes in mass are caused by imbalance in the fluxes into and out of the volume

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot (\mathbf{u}\rho)$$

We can apply the same arguments to momentum, except that forces can act as sources or sinks. Thus the momentum in the x direction, ρu, will satisfy

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \mathbf{u}) = F_1$$

or

$$\frac{\partial}{\partial t}u + \mathbf{u} \cdot \nabla u = \frac{D}{Dt}u = \frac{1}{\rho}F_1$$

where $\mathbf{F}$ is the force acting per unit volume at the Eulerian location $\mathbf{x}$. We point out one important caveat:

- This equation is only valid in Cartesian coordinates. If we denominate u as the eastward velocity on a spherical earth, then a particle moving at a constant speed traverses a circle (line of constant latitude), and its velocity vector is changing with time – it is accelerating. Accordingly, in a longitude–latitude–height system, the nonlinear term in the acceleration must be more complex than the dot product of $\mathbf{u}$ with the gradient of each component. We shall examine the proper form in chapter x.x; for now, Cartesian geometry is adequate. and we can use the form above.

1.4.1 — Forces

The forces acting on a rotating stratified fluid are gravity (which appears as buoyancy forces), pressure, Coriolis, and viscous stresses. We need to represent each of these as the force exerted per unit volume.

GRAVITY: The effects of gravity are straight-forward: the force is g times the mass in the downward or negative z direction. The force per unit volume is

$$\mathbf{F} = -\rho g \hat{\mathbf{z}}$$

CORIOLIS "forces" act on matter moving in a rotating system. Suppose we consider three snapshots of a particle subject to no external forces viewed in both a fixed (inertial) and a rotating frame of reference (figure 1.13).

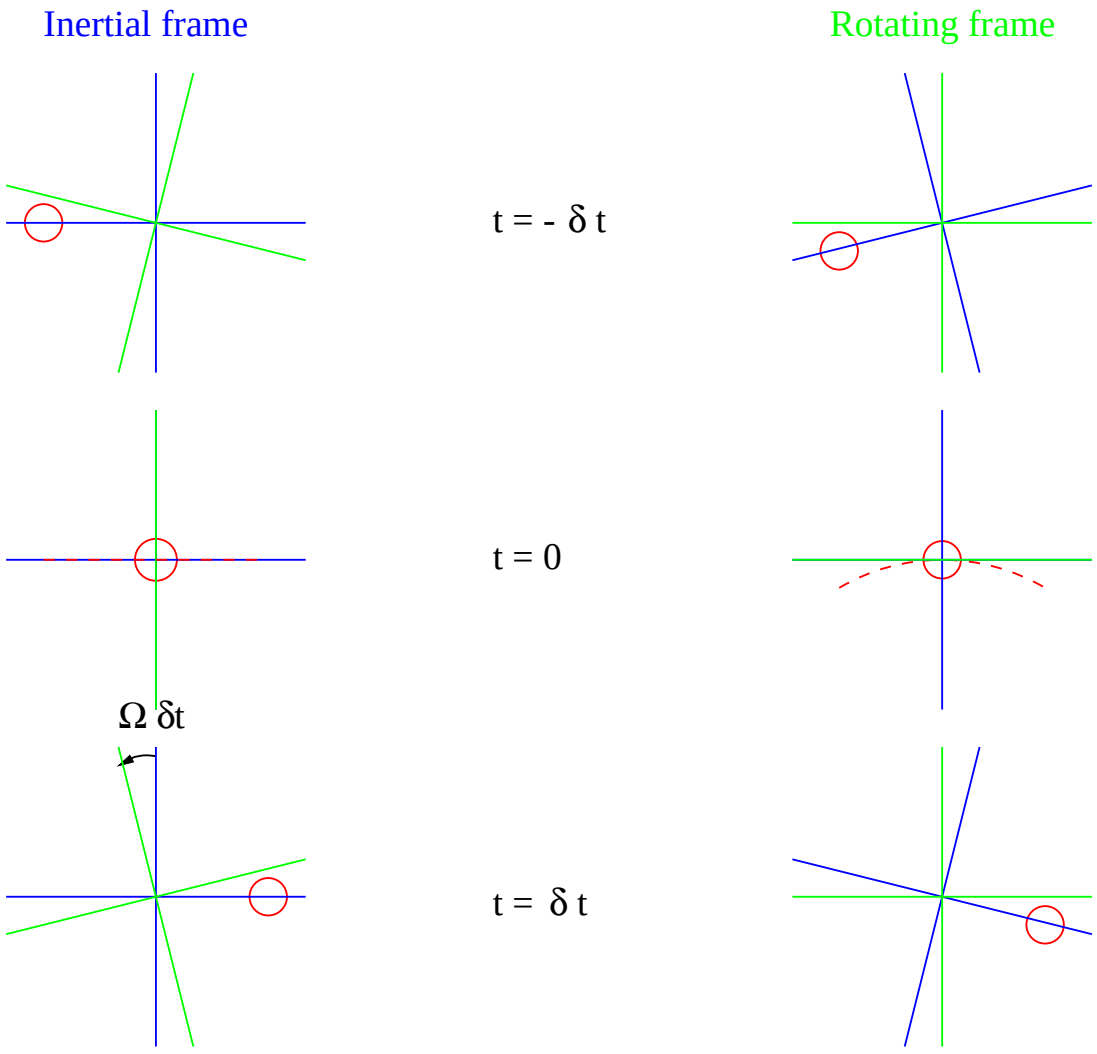

Figure 1.13: Particle positions in fixed and rotating frame. Blue lines show the coordinate axes in the fixed frame, green lines in the rotating frame.

In inertial space , the particle is moving in a straight line; we set $t = 0$ as the time when it passes through the origin heading along the x-axis. In the inertial (fixed) frame, its position is given by

$$\mathbf{x}_f = (u_0 t, 0, 0)$$

giving successive points

$$\mathbf{x}_f : \ (-u_0\delta t, 0, 0) \to (0, 0, 0) \to (u_0\delta t, 0, 0)$$

Correspondingly, the positions in the rotating frame are

$$\mathbf{x} : \ (-u_0\delta t\cos(\Omega\delta t), -u_0\delta t\sin(\Omega\delta t), 0) \to (0, 0, 0) \to$$

$$(u_0\delta t\cos(\Omega\delta t), -u_0\delta t\sin(\Omega\delta t), 0)$$

where Ω is the rotation rate of the reference frame.

Clearly the particle accelerates in the $-y$ direction. Indeed, for this case, using an approximation to the second derivative gives

$$\begin{aligned}
\frac{d^2\mathbf{x}}{dt^2} &\simeq \frac{[\mathbf{x}(t+\delta t) - \mathbf{x}(t)] - [\mathbf{x}(t) - \mathbf{x}(t-\delta t)]}{\delta t^2} \\
&= \frac{\mathbf{x}(t+\delta t) + \mathbf{x}(t-\delta t) - 2\mathbf{x}(t)}{\delta t^2} \\
&= \frac{(0, -2u_0\delta t\sin(\Omega\delta t), 0)}{\delta t^2} \\
&= -2\Omega u_0\hat{\mathbf{y}} = -2\vec{\Omega}\times\mathbf{u}
\end{aligned}$$

Applying the same argument to a particle moving north shows that it also accelerates to the right.

If we were to postulate some force as causing this acceleration, the strength would be

$$\mathbf{F} = -\rho 2\vec{\Omega}\times\mathbf{u}$$

This Coriolis "force" is of course an artifact of dealing with movement in an accelerating reference frame (remember that circular motion has a velocity vector which is constantly changing with time) but it can be used just as though it were real.† Usually, however, we will put this term on the left-hand side to keep it with the accelerations relative to the earth

$$\frac{D}{Dt}\mathbf{u} + 2\vec{\Omega}\times\mathbf{u} = -g\hat{\mathbf{z}} + \frac{1}{\rho}\mathbf{F}$$

with F being the remaining two forces.

† One tip-off that the "force" is an artifact of the reference frame definition is its proportionality to mass (unlike, for example, electrical forces). In general relativity, gravity too is associated with the reference frame characteristics (its curvature), so that the force likewise is proportional to the mass.

PRESSURE represents the forces that the molecules exert as they bounce off each other during their thermal fluctuations (not the average velocity $\mathbf{u}$). Conceptually, if we consider a wall in a fluid with no average motion, each time a molecule bounces off a wall, it applies a force to the wall (and the wall applies an equal and opposite force to reverse the normal component of the molecule's velocity). The net force is the product of the average normal velocity, the mass of the molecules, and the number hitting the wall per unit time. If we double the size of the wall, we double the number of molecules impinging on it, and double the force. To account for this, we define the pressure as the force per unit area.

Now consider the forces on a small cube-shaped object centered on location x in the fluid. If the thermal motion is the same everywhere in the fluid, the forces exerted on the box by molecules bouncing off the left wall will be equal and opposite to that exerted by molecules bouncing off the right wall. Therefore the net force on the cube will be zero. But if the speeds of the molecules on the right are higher than that of those on the left, the force on the right side of the box pushing it to the left will be greater than the force on the left side pushing it to the right. The non-zero net force depends on changes in pressure and will try to push the box towards the lower pressure regions. The same argument applies if we replace the solid box with a parcel of fluid; if the molecules on the right are moving faster, collisions with them will apply more force on the fluid parcel than those with the molecules on the left. Thus we can see that the force depends on the gradient of the pressure.

To formalize this, we use the definition of pressure, as the normal force per unit area exerted by fluid outside a volume on the fluid inside, to write

$$\mathbf{F}V = \int_{\partial V} -p\hat{\mathbf{n}} d^2\mathbf{x} \quad \Rightarrow \quad F_1 = -\frac{1}{V}\int_{\partial V} p\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}\, d^2\mathbf{x} = -\frac{1}{V}\int_V \nabla\cdot(\hat{\mathbf{x}}p)\, d^3\mathbf{x}$$

In the limit, the force per unit volume is

$$F_1 = -\frac{\partial p}{\partial x} \quad \Rightarrow \quad \mathbf{F} = -\nabla p$$

VISCOUS STRESSES are tangential forces acting across a surface; conceptually, a faster moving (on average) eastward stream located (for example) to the north of a slower stream will impart some of its momentum to the slower stream by collisions between the molecules, in effect exerting an eastward force. The slower stream has the opposite effect on the faster one. Thus, the tendency is to equalize the velocities; the stresses act much like diffusion of velocity

$$\mathbf{F} = \rho\nu\nabla^2\mathbf{u}$$

where ν is the kinematic viscosity having units (like diffusivity) of L^2/T.

MOMENTUM EQUATIONS: Putting all the forces together gives the momentum equations

$$\frac{D}{Dt}\mathbf{u} + 2\vec{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\hat{\mathbf{z}} \tag{1.14}$$

1.4.2 — Thermodynamics, buoyancy, and the Boussinesq approximation

The momentum and mass equations are not sufficient to predict the evolution of the flow: given the current state at time t, we know how $\mathbf{u}$ and ρ change with time but cannot determine p at $t + \delta t$. Fluids have an **equation of state** relating the density to other properties including the pressure; for seawater, this is expressed as

$$\rho = \rho(S, T, p)$$

where S is the salinity (grams of salt per kilogram of seawater) and T is the temperature. If ρ were only a function of pressure, we could invert the relationship to find the new pressure given the new density; however, the dependence on T and S implies we need two additional evolution equations.

For simplicity, we shall avoid these complications and make the Boussinesq approximation (see appendix xx for details). We let

$$\rho \equiv \rho_0(p)\left(1 - \frac{B}{g}\right)$$

The variable $B = g\frac{\rho_0 - \rho}{\rho_0}$ represents the buoyant acceleration, upwards when the density is lower than average and downwards when it is higher; in the fluid, the effects of gravity are much reduced – most of it is compensated for by pressure forces (figure 1.14).

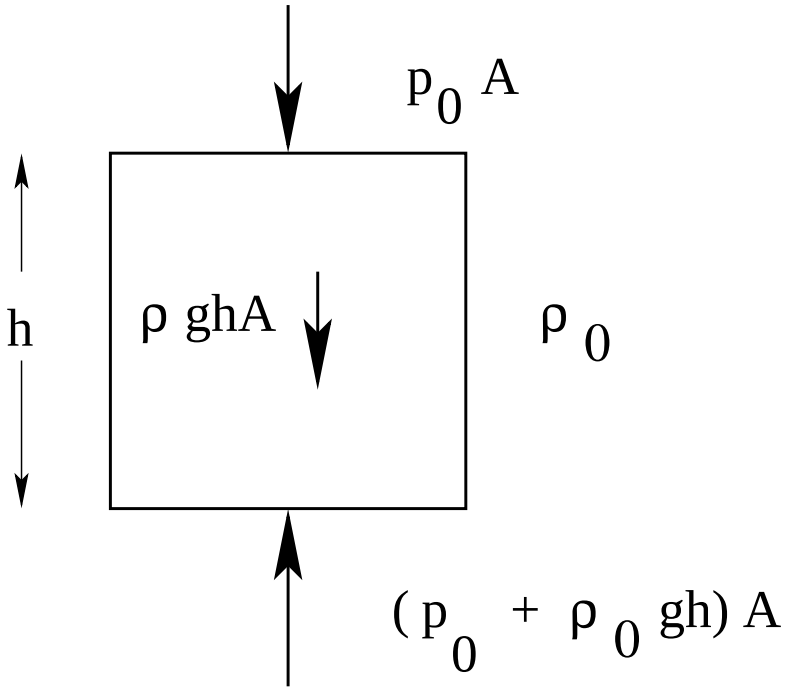

Figure 1.15: The net force per unit mass is $(-p_0 A + p_0 A + \rho_0 g h A - \rho g h A)/\rho h A = g(\rho_0 - \rho)/\rho$.

The $\rho_0(p)$ takes into account the most significant part of the compressibility of sea water, the overall increase in density with depth (figure 1.14).

We also define a pressure-like quantity ϕ such that

$$\nabla\phi = \frac{1}{\rho_0}\nabla p + g\hat{\mathbf{z}}$$

so that the pressure gradient and gravitational terms become

$$\begin{aligned} -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} &= -\frac{1}{1-B/g}\nabla\phi + g\left(\frac{1}{1-B/g} - 1\right)\hat{\mathbf{z}} \\ &= -\frac{1}{1-B/g}\nabla\phi + \frac{B}{1-B/g}\hat{\mathbf{z}} \\ &\simeq -\nabla\phi + B\hat{\mathbf{z}} \end{aligned}$$

where the last step assumes that B is small compared to g.

The thermodynamic and salinity equations give

$$\frac{D}{Dt}B = \kappa\nabla^2 B + \mathcal{H}$$

where $\mathcal{H}$ represents buoyancy sources from heating or freshening. We've assumed that (1) both the flow speed and $\sqrt{gH}$ (the long surface wave speed) are small compared to the sound speed and (2) κ repesents small scale mixing which transfers heat and salt similarly rather than the molecular processes which give quite different diffusivities.

Neglecting terms of similar order in the mass conservation equation show that the flow is nearly non-divergent. Putting these equations together gives the Boussinesq system:

$$\begin{aligned} \frac{D}{Dt}\mathbf{u} + 2\vec{\Omega}\times\mathbf{u} &= -\nabla\phi + B\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0 \\ \frac{D}{Dt}B &= \kappa\nabla^2 B + \mathcal{H} \end{aligned} \qquad (Bouss)$$

At first sight, we have returned to the dilemma of not having a predictive equation for ϕ; however, we can diagnose its value at any time by requiring that the flow remain non-divergent; i.e., $\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u}) = 0$. This implies that

$$\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - 2\vec{\Omega}\cdot\boldsymbol{\zeta} = -\nabla^2\phi + \frac{\partial B}{\partial z}$$

which can be solved to give the pressure given the velocity and buoyancy. The curl of the velocity

$$\boldsymbol{\zeta} = \nabla\times\mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

is called the **vorticity** and plays a central role in the dynamics of geophysical flows. It measures the swirl of the fluid around each axis; to see this relationship, consider the flow around some imaginary circuit drawn in the fluid

$$\mathcal{C} \equiv \oint \mathbf{u} \cdot d\vec{\ell}$$

By Stokes' theorem, this measure of the rate of swirling around some axis, called the circulation, is equal to the integral of the vorticity normal to the circuit, i.e., along the axis

$$\mathcal{C} = \iint \boldsymbol{\zeta} \cdot \hat{\mathbf{n}} \, dA$$

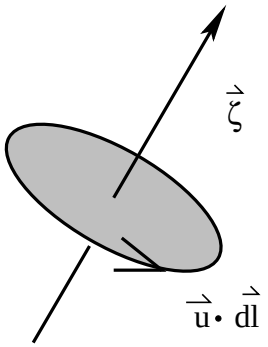

Figure 1.16: Circulation around an axis

In the upwelling example (figure 1.1), we will have vorticity directed along the y axis.

1.4.3 — Upwelling theory – linear version

As a first model of the flow, we shall make a number of simplifications (some of which will be remedied as we build a numerical model for the flow):

1) Straight coast: the topography, wind, and all flow variables are independent of the alongshore distance ($\frac{\partial}{\partial y} = 0$, c.f. Allen, 19xx). This kind of idealization is very useful, since it reduces a three dimensional problem to a two dimensional one. Yet it can be misleading; in the presence of strong alongshore currents, even small amplitude or large-scale downstream variations may be significant (Chapman, 19xx)
2) Weak flows: we take advection to be much smaller than the Coriolis, pressure gradient, and viscous terms. The Coriolis term will be written using only the vertical component of the rotation $2\vec{\Omega} = f\hat{\mathbf{z}}$. In the buoyancy equation, we assume that the deviations $B'(\mathbf{x}, t) = B(\mathbf{x}, t) - \overline{B}(z)$ from a stratified state are small and neglect advection of B' but not of $\overline{B}$. Likewise we split ϕ into $\overline{\phi} + \phi'$.
3) Vertical mixing: we presume that the dominant mixing is turbulence acting on the more rapid vertical variations of properties and neglect horizonal mixing.

Scale analysis

We can use the approach of scale analysis to decide the conditions under which such approximations might be valid. We define "scales" for the different variables: for the dependent variables, the scale value U, for example, would be the characteristic magnitude of u; for independent variables, the scales represent the characteristic magnitude of some field divided by the characteristic magnitude of its derivative so that we replace $\frac{\partial}{\partial x}$ by $1/L$ in estimating sizes of the different terms in an equation. We can then decide which terms are small compared to the others and drop them from subsequent analysis.

For example, mass equation and the scales of terms looks like

$$\begin{array}{ccc} \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial z}w = 0 \\ \dfrac{U}{L} \qquad \dfrac{W}{H_\infty} \end{array}$$

where the horizontal and vertical scales are set by the topographic profile

$$H = H_\infty \tanh(x/L)$$

The last equation makes it clear that the use of mathematical functions necessitates specifying scales: we cannot define the hyperbolic tangent of a dimensional length such as 1.5 m. Trying to say it is tanh(1.5) as defined by a table or calculator does not work; would one then say that tanh of 1500 cm is tanh(1500)? [†] Rather, we always deal with functions acting on non-dimensional numbers which are the ratios of dimensional variables such as x to scales with the same dimensions, L. But extending the argument a little further, we could also be dealing with a shelf-slope topography $H = 10\ m \tanh(x/3\ km) + 500\ m \tanh([x - 150\ km]/50\ km) + 500\ m$ which has multiple scales for depth and length; therefore, scale analysis serves as a guide, but results should be checked *a-posteriori* to verify that neglected terms are indeed unimportant.

For the mass equation, however, we will not drop terms since that would leave us with an equation such as $\frac{\partial u}{\partial x} = 0$ which has only trivial solutions; instead, we use the scale analysis to find the sizes of terms for which we may not have an estimate for (such as W). Thus, we expect $W = UH/L$.

[†] To view the problem another way, note that we often define functions as a series: $\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} \ldots$. If we tried to substitute 1.5 m into this formula, we'd have to add 1.5 m, 1.125 m^3, and 1.0125 m^5, which does not make physical sense.

The momentum equations

$$\begin{array}{ccccccc}
\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + w\frac{\partial}{\partial z}u - fv = -\frac{\partial}{\partial x}\phi + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial z^2} \\
\frac{U}{T} \quad \frac{U^2}{L} \quad \frac{WU}{H_\infty} \quad fV \quad \frac{\Phi}{L} \quad \frac{\nu U}{L^2} \quad \frac{\nu U}{H_\infty^2} \\
\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + w\frac{\partial}{\partial z}v + fu = \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial z^2} \\
\frac{V}{T} \quad \frac{UV}{L} \quad \frac{WV}{H_\infty} \quad fU \quad \frac{\nu V}{L^2} \quad \frac{\nu V}{H_\infty^2} \\
\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + w\frac{\partial}{\partial z}w = -\frac{\partial}{\partial z}\phi + B' + \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial z^2} \\
\frac{W}{T} \quad \frac{UW}{L} \quad \frac{W^2}{H_\infty} \quad \frac{\Phi}{H} \quad B' \quad \frac{\nu W}{L^2} \quad \frac{\nu W}{H_\infty^2}
\end{array}$$

suggest choosing $\Phi \sim fVL$, $U \sim \frac{\nu}{fH_\infty^2}V$, $B' \sim fVL/H_\infty$, $T \sim L/U$. Using these and normalizing the equations by the Coriolis or pressure terms gives the relative sizes:

$$\begin{array}{ccccc}
\frac{D}{Dt}u - fv = -\frac{\partial}{\partial x}\phi + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial z^2} \\
\epsilon E^2 \quad 1 \quad 1 \quad \delta^2 E^2 \quad E^2 \\
\frac{D}{Dt}v + fu = \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial z^2} \\
\epsilon \quad 1 \quad \delta^2 \quad 1 \\
\frac{D}{Dt}w = -\frac{\partial}{\partial z}\phi + B' + \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial z^2} \\
\delta^2\epsilon E^2 \quad 1 \quad 1 \quad \delta^4 E^2 \quad \delta^2 E^2
\end{array}$$

where the non-dimensional parameters characterizing the flow are

$$\begin{aligned}
\text{Rossby number} \quad & \epsilon = \frac{V}{fL} \\
\text{Aspect ratio} \quad & \delta = \frac{H_\infty}{L} \\
\text{Ekman number} \quad & E = \frac{\nu}{fH_\infty^2}
\end{aligned}$$

These represent, respectively, the strength of advection (compared to Coriolis accelerations), the geometric constraint (which reduces the ratio of vertical to horizontal velocities), and the strength of friction (again compared to the Coriolis term). For the model we are now considering, we assume that all of these are small. Note that we still need to relate V to the external forcing; however, we can reduce the wind forcing until the Rossby number is indeed smaller than 1.

Over most of the flow, two important balances hold:

- The Coriolis force associated with the alongshore current is compensated by the cross-shelf pressure gradient; the near-equality of these forces is called **geostrophic balance** and applies in many larger scale flows, where it takes the more general form

$$f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\phi' \quad \Rightarrow \quad \mathbf{u} = \frac{1}{f}\hat{\mathbf{z}} \times \nabla\phi'$$

  In ocean eddies, as in atmospheric weather systems, the fluid moves along the lines of constant pressure, rather than accelerating down the gradient. In the northern hemisphere, the flow will have the high pressure to the right, and the speed will be proportional to the gradient (i.e., inversely proportional to the spacing between pressure contours – see figure 1.16).
- Vertically, the fluid remains in **hydrostatic balance**

$$\frac{\partial}{\partial z}\phi' = B'$$

  so that we can find the pressure by integrating the density field. However, this process introduces an unknown function of x, y, and t at the depth where the integration begins.

Now we examine the buoyancy equation

$$\frac{D}{Dt}B' + wN^2 = \kappa\frac{\partial^2}{\partial x^2}B' + \kappa\frac{\partial^2}{\partial z^2}B'$$

$\frac{UB'}{L}$	WN^2	$\frac{\kappa B'}{L^2}$	$\frac{\kappa B'}{H_\infty^2}$
ϵ	S	$\frac{\delta^2}{Pr}$	$\frac{1}{Pr}$

where $N^2 \equiv \frac{\partial}{\partial z}\overline{B}$ is square of the Brunt-Väisälä frequency. I.e., if we lift a blob of fluid in a stratified system, it is heavier than its surroundings (negatively buoyant) and accelerates downwards. It passes the initial position, decelerates, and comes to rest below, where it now feels a positive buoyancy force. The period of the resulting oscillation is $2\pi/N$.

The additional parameters above are

$$\text{Stratification} \qquad S = \frac{N^2 H_\infty^2}{f^2 L^2}$$

$$\text{Prandtl number} \qquad Pr = \frac{\nu}{\kappa}$$

and are both assumed to be order one.

Thus we arrive at a simplified dynamics holding over most of the fluid:

$$\begin{aligned} fv &= \frac{\partial}{\partial x}\phi' \\ B' &= \frac{\partial}{\partial z}\phi' \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\ fu &= \nu \frac{\partial^2 v}{\partial z^2} \\ N^2 w &= \kappa \frac{\partial^2 B'}{\partial z^2} \end{aligned} \tag{1.10}$$

We can eliminate variables from these equations to get a single PDE for ϕ or for w; however, it is most convenient to work in terms of a streamfunction ψ for the flow.

Streamfunctions

For a two-dimensional incompressible flow, we can define the streamfunction difference between two points $\psi(\mathbf{x}_2) - \psi(x_1)$ as the volume of fluid passing through a surface formed by a curve joining the two points and sweeping a unit distance in the third direction. Because the flow in non-divergent, we will get the same answer for any curve joining the two points as long as it can be deformed into the original curve without crossing any obstacles in the flow. From this definition, we have

$$\begin{aligned} \psi(\mathbf{x}_2 + \delta x \hat{\mathbf{x}}) - \psi(\mathbf{x}_2) &= \delta x \frac{\partial \psi}{\partial x} = -w(\mathbf{x}_2)\delta x \\ \psi(\mathbf{x}_2 + \delta z \hat{\mathbf{z}}) - \psi(\mathbf{x}_2) &= \delta z \frac{\partial \psi}{\partial z} = u(\mathbf{x}_2)\delta z \end{aligned}$$

so that

$$u = \frac{\partial \psi}{\partial z} \quad , \quad w = -\frac{\partial \psi}{\partial x} \quad or \quad \mathbf{u} = -\nabla \times \psi \hat{\mathbf{y}}$$

which clearly satisfies the mass equation $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$. The streamfunction acts much like the pressure in a geostrophic flow: instantaneously, the velocity vector is tangent to the contours of ψ, satisfying a "left-hand rule" (because of the negative sign), and the speed is inversely proportional to the separation of the ψ contours.

The upwelling equations can be reduced to an equation for ψ by first eliminating ϕ from the geostrophic and hydrostatic equations

$$f \frac{\partial v}{\partial z} = \frac{\partial B'}{\partial x}$$

(known as the thermal wind equation). The last two equations become

$$f \frac{\partial \psi}{\partial z} = \frac{\nu}{f} \frac{\partial^2 B'}{\partial x \partial z} \quad , \quad -N^2 \frac{\partial \psi}{\partial x} = \kappa \frac{\partial^2 B'}{\partial z^2}$$

from which we can eliminate the buoyancy to derive

$$\frac{f^2}{\nu} \frac{\partial^2 \psi}{\partial z^2} + \frac{N^2}{\kappa} \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{1.11}$$

Boundary conditions

Now we must consider the boundary conditions to apply when solving eqn. 1.13. The flow normal to the bottom must be zero; this implies ψ should be constant ($\psi = 0$ since we can add an arbitrary constant to the streamfunction without affecting the flow) along the bottom. We might think that the same argument applies at the top, and the constant must be the same since $\int_{-H}^{0} dz u(x,z) = 0 = \psi(x,0) - \psi(x,-H)$; however, ψ would then be zero everywhere. What has gone wrong? The problem is that the full boundary conditions at the top are

$$w = 0 \quad , \quad \nu \frac{\partial v}{\partial z} = \frac{\tau}{\rho_s} \quad , \quad \nu \frac{\partial u}{\partial z} = 0 \quad at \quad z = 0$$

where τ is the wind stress and ρ_s the surface density. Our simplified system, eqns 1.11, cannot satisfy all of these simultaneously. There is a thin region near the surface with a characteristic scale $h_{ek} = \sqrt{\nu/f}$ which has an associated Ekman number of 1; in this region (the Ekman layer) other terms appear in the dynamics and permit us to match all of the conditions.

We can get at this directly by dropping only the terms which are order δ^2 or ϵ and then forming a streamfunction equation. The result

$$\frac{f^2}{\nu}\frac{\partial^2 \psi}{\partial z^2} + \frac{N^2}{\kappa}\frac{\partial^2 \psi}{\partial x^2} + \nu \frac{\partial^6 \psi}{\partial z^6} = 0$$

allows specifying three boundary conditions at the top and at the bottom (the no-normal flow plus two stress conditions at the top, and vanishing normal and tangential flow at the bottom).

However, we can gain more insight by examining the boundary layer flow directly; we define the velocities near the surface (shallower than $-h_{ek}$) to be those predicted by upwelling equations plus a correction $\mathbf{u}_{ek}$. These velocities vanish at depths $z \ll h_{ek}$. Since the upwelling equations define the velocities in terms of the pressure, we do not need a correction to ϕ. (This is related to the fact that solving the diagnostic equation for pressure will smooth out small scale structure in the velocities.) The correction equations

$$\begin{aligned} -f v_{ek} &= \nu \frac{\partial^2}{\partial z^2} u_{ek} \\ f u_{ek} &= \nu \frac{\partial^2}{\partial z^2} v_{ek} \\ \frac{\partial u_{ek}}{\partial x} + \frac{\partial v_{ek}}{\partial y} + \frac{\partial w_{ek}}{\partial z} &= 0 \end{aligned}$$

can be integrated vertically to see that the horizontal transports in the surface layer are related to the wind stress

$$\begin{aligned} f \int v_{ek} &= -\tau^{(x)}/\rho_s \\ f \int u_{ek} &= \tau^{(y)}/\rho_s \\ \frac{\partial}{\partial x}\int u_{ek} + \frac{\partial}{\partial y}\int v_{ek} + w_{ek}(0) &= 0 \end{aligned}$$

These take care of the two stress conditions; satisfying the no normal flow condition gives

$$w_{ek}(0) + w(0) = 0 \quad \Rightarrow \quad w(0) = \frac{\partial}{\partial x}\frac{\tau^{(y)}}{f\rho_s} - \frac{\partial}{\partial y}\frac{\tau^{(x)}}{f\rho_s}$$

For the upwelling problem, we have an offshore transport in the surface layer (for $\tau = \tau^{(y)} > 0$), and, if the wind increases offshore, that transport also increases. To provide this additional fluid, water must upwell from below. The effective boundary condition for the interior flow is

$$w = -\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\frac{\tau}{f\rho_s} \quad or \quad \psi = -\frac{\tau}{f\rho_s}$$

We take

$$\tau = \tau_0(1 - \exp(-x/L_\tau)$$

to avoid a singularity at the coast. To summarize, we must solve

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z^2} + \frac{N^2\nu}{f^2\kappa}\frac{\partial^2\psi}{\partial x^2} &= 0 \\ \psi = 0 \quad at \quad z &= -H_\infty \tanh(x/L) \\ \psi = -\frac{\tau_0}{f\rho_s}(1 - \exp(-x/L_\tau) \quad at \quad z &= 0 \end{aligned} \tag{1.12}$$

The solution is determined by two parameters $N^2H_\infty^2\nu/f^2L^2\kappa = S\ Pr$ and L_τ/L. The idealized flow used previously corresponds to the weakly stratified solution when $S\,Pr << 1$, so that equation (1.13) can be approximated by

$$\frac{\partial^2\psi}{\partial z^2} \simeq 0$$

The streamfunction just linearly interpolates between the value at the base of the surface layer and the zero value at $z = -H_\infty \tanh(x/L)$. As we increase the strength of the stratification, the streamfunction spreads out horizontally (figure 1.17): the stronger stratification inhibits the vertical motion. The x scale where horizontal and vertical derivative are comparable is $R_d \sim NH/f$; it gives a rough measure of the width over which w will be significant.

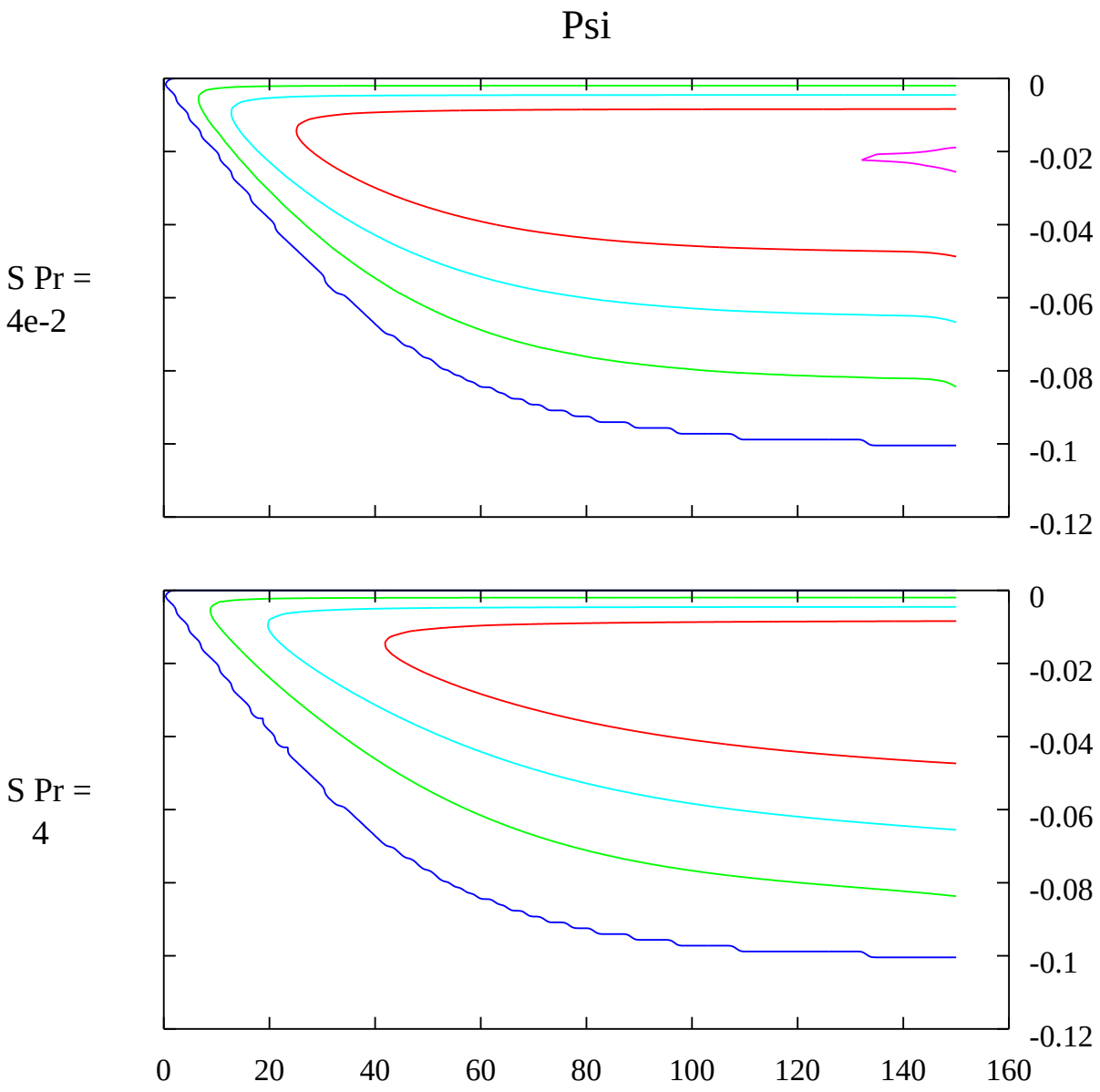

Figure 1.17: Contours of ψ for $S\,Pr = 4 \times 10^{-2}$ (standard case) and $S\,Pr = 4$

1.5 — Putting it all together

Adding both advection and diffusion gives a picture which is similar to the Lagrangian view but also differs in significant ways. The full equations are

$$\frac{D}{Dt}P = P[\mu(N_t - P - Z) - gZ - {}_P] + \nabla_h \kappa_h \nabla_h P + \frac{\partial}{\partial z}\kappa_z \frac{\partial}{\partial z}P$$
$$\frac{D}{Dt}Z = Z[agP - d_Z] + \nabla_h \kappa_h \nabla_h Z + \frac{\partial}{\partial z}\kappa_z \frac{\partial}{\partial z}Z$$

where the horizontal mixing rates are presume to be different from the vertical ones. This is a common approach, but, as we shall discuss later is rather dubious – we might expect

the difference in rates to arise with respect to mixing along vs. across density surfaces instead of level surfaces. For the upwelling problem, the vertical mixing dominates, so the assumption is probably not significant. (The numerics may end up giving substantial artificial diffusion as well.)

To solve this, we deal with the Eulerian equations on an x–y grid; the advective and diffusive fluxes through the boundaries of the grid boxes are approximated numerically, and the values of P and Z in the grid boxes are stepped forward in time. We discuss numerics briefly in Appendix xx; there are many difficulties associated with numerical solution of advection-diffusion-reaction equations. An ideal advection-diffusion scheme would

- conserve tracers: these processes only move material around without changing the total amount.
- maintain positive values and appropriate magnitudes of local maxima: advection by itself preserves maximum and minimum values; diffusion reduces the strength of extrema. In particular, it is important that the biological fields remain positive. Negative values are not only meaningless, but, if large enough can cause exponential growth in the fields. In practice, this requirement is difficult to achieve, and we may have to tolerate small negative values.
- have small phase error, so that peaks move at the proper speed (either by advection or wave propagation).
- have minimal artificial (numerical) diffusion.
- handle complex topography and coastlines.

These are difficult requirements, and existing numerical schemes end up compromising among them. We use a simple scheme (appendix xx) to examine the joint effects of advection, diffusion, and biological dynamics. We have specified the streamfunction from the previous section and taken the diffusivities to be $\kappa_h = 0.1\ km^2/d = 1.2\ m^2/s$ for the horizontal direction and $\kappa_v = 10^{-6}\ km^2/d = 1.2 \times 10^{-5}\ m^2/s$.

The biological fields (figure 1.18) look quite similar to the Lagrangian picture, except that the peak values are reduced by the mixing and the double maxima structure in P disappears. The joint effects of advection and diffusion show more strikingly in the case of a deeper shelf ($200\ m$). Without mixing or advection, organisms cannot survive in the deep water (below $97\ m$). When upwelling begins, a band of water with high nutrients will be brought to the surface, but, since it has no living biota, it remains at $N = N_T$, $P = Z = 0$ as it spreads out along the surface (figure 1.18c-d). In contrast, when mixing occurs, the deep water is not completely lifeless, and the high nutrient water which is upwelling is also seeded with phytoplankton from the surroundings. As a result, the phytoplankton maximum is at the surface near the coast, whereas for the non-mixing case it occurs deeper and further offshore where the water with $P > 0$ but $Z = 0$ is upwelling. But we can also see a tongue of relatively low values extending up along the topography from below the euphotic zone. The zooplankton field shows more clearly the upwelling of the very low Z water and the gradual repopulation as it travels along the surface and mixes with the water at the base of the the Ekman layer. Because of the very small Z values in the corner, the phytoplankton are able to take up nearly all the nutrient and attain a higher population density than in the shallower experiment where grazing pressure remains everywhere significant.

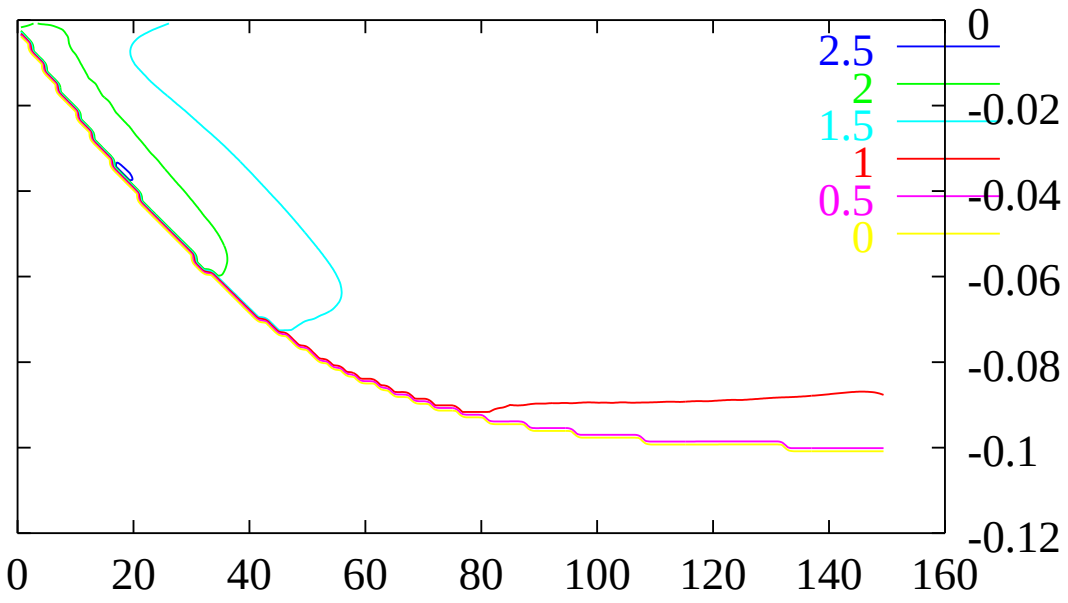

Figure 1.18a: Phytoplankton densities, shelf depth=100 m

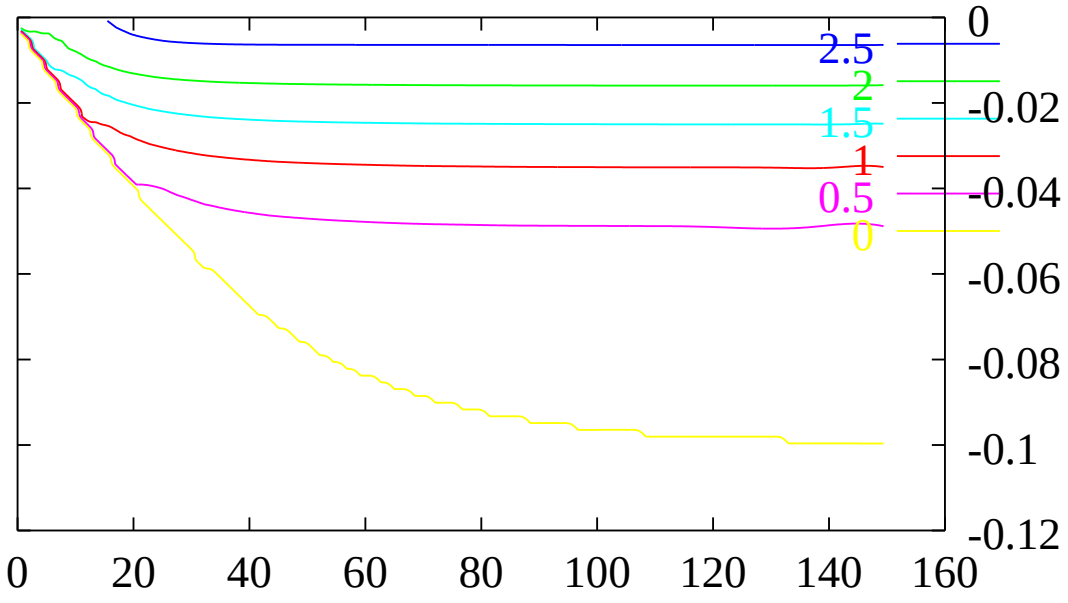

Figure 1.18b: Zooplankton densities

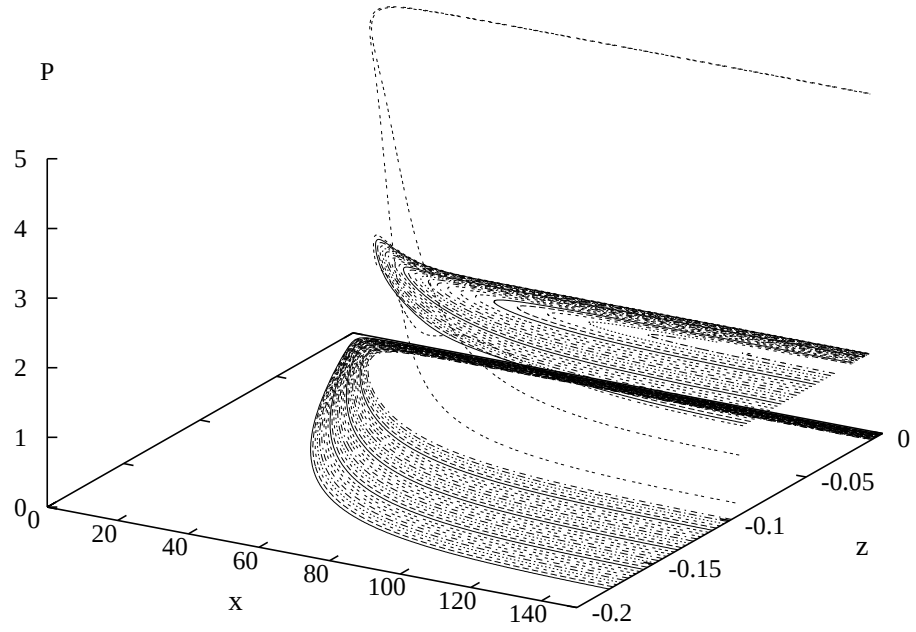

Figure 1.18c: Phytoplankton biomass along trajectories.

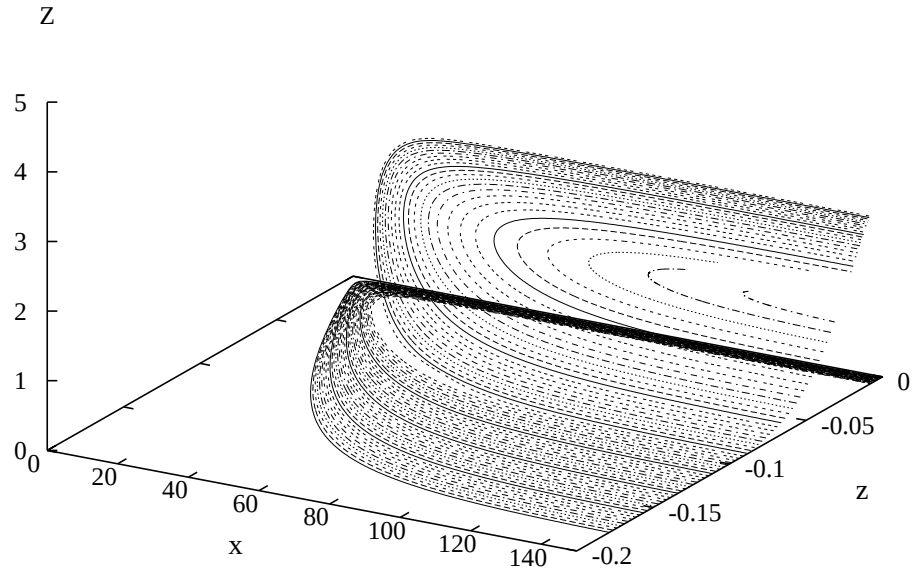

Figure 1.18d: Zooplankton biomass along trajectories.

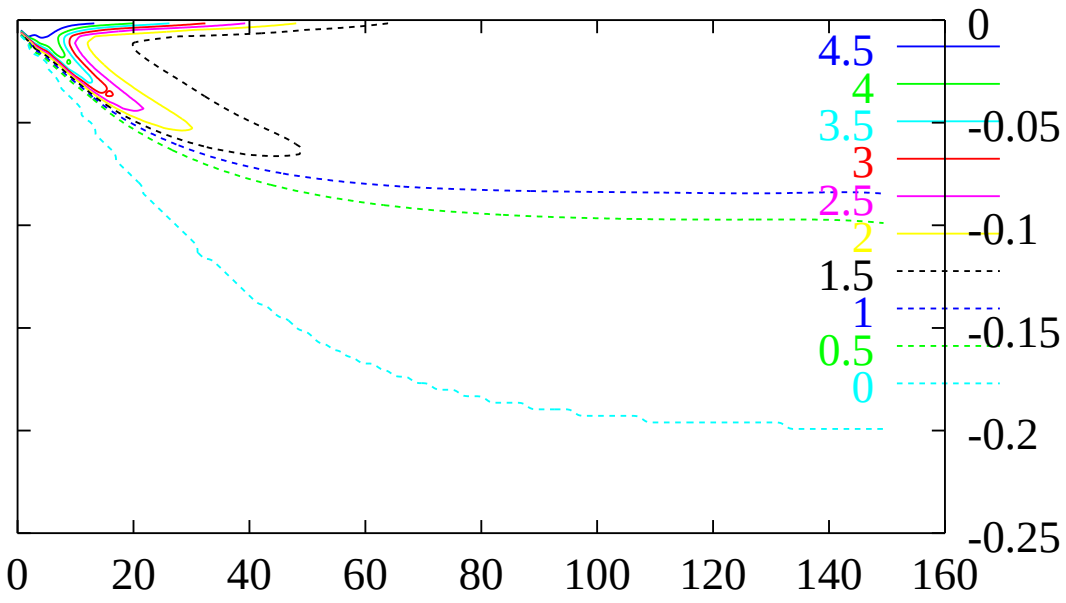

Figure 1.18e: Phytoplankton densities, shelf depth=200 m

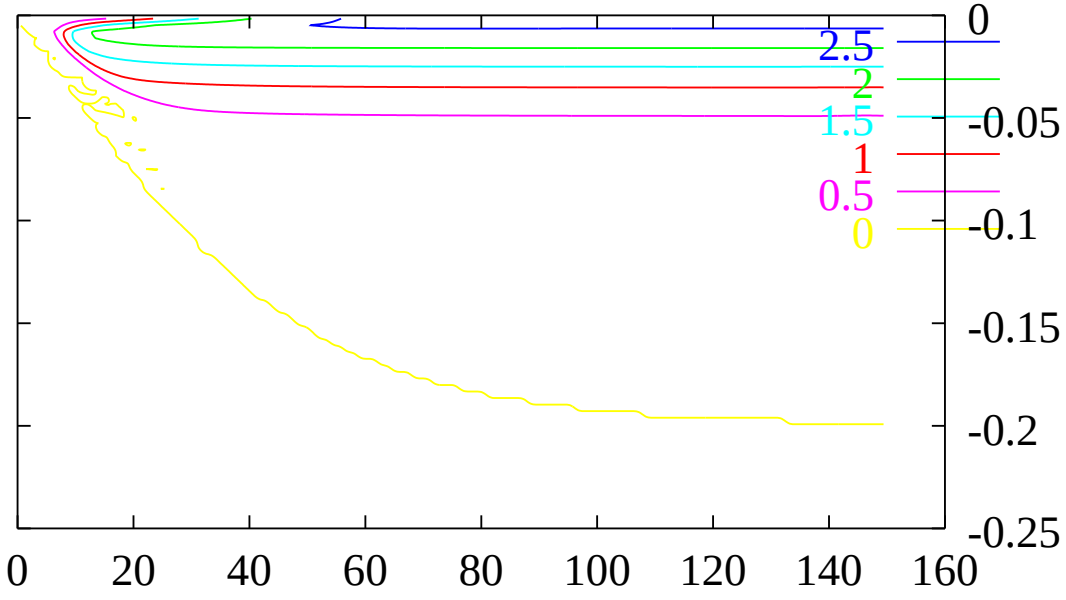

Figure 1.18f: Zooplankton densities

1.5.1 — Time-dependence

Although the solutions we have discussed so far do exhibit time-dependence as the biology comes into equilibrium (including a weak bloom in the shallower case), we cannot expect them to capture the variability associated with a sudden wind change since we have not considered the transients in the physics. For the time-dependent flow model, we shall use the hydrostatic, Boussinesq equations which form the basis for most mesoscale to large scale ocean models. (Investigating some processes, such as convection, does require including the vertical acceleration terms.)

$$\begin{aligned}
\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}}\times\mathbf{u} &= \nabla\phi + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\mathbf{u} \\
\frac{\partial\phi}{\partial z} &= B \\
\nabla\cdot\mathbf{u} + \frac{\partial w}{\partial z} &= 0 \\
\frac{D}{Dt}B &= \frac{\partial}{\partial z}\kappa\frac{\partial}{\partial z}B \\
\frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + w\frac{\partial}{\partial z}
\end{aligned}$$

where $\mathbf{u}$ now represents just the horizontal velocities and ∇ the gradient or divergence in the horizontal directions only.

Appendix xx discusses the general procedure for solving the hydrostatic model. (In addition, it demonstrates that taking the vertical coordinate to be pressure – or, more precisely, $-p/\rho_0$ – leads to the same set of equations without requiring the density variations to be small. See deSzoeka and Samelson, 2002.) Essentially, we compute the pressure from the density field by integrating the hydrostatic relation, with the surface pressure (or the free-surface elevation) serving as an unknown integration constant at each x location. We calculate

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\mathbf{u}\cdot\nabla u + fv - \frac{\partial}{\partial x}\int_0^z dz'\; B(z') - \frac{\partial}{\partial x}\phi_s(x) \\
&= R_u - \frac{\partial}{\partial x}\phi_s(x)
\end{aligned}$$

Mass conservation implies

$$\int_{-H}^{0} dz\, u(z) = 0 \quad \Rightarrow \quad \int_{-H}^{0} dz\, \frac{\partial u(z)}{\partial t} = 0$$

so that the surface pressure gradient is simply the vertical average of the part of the acceleration terms, R_u, which are known from the current velocity and buoyancy state:

$$\frac{\partial}{\partial x}\phi_s = \frac{1}{H}\int_{-H}^{0} R_u$$

Practically, then, we calculate $\frac{\partial}{\partial t}u$ ignoring the surface pressure contribution and then subtract off the vertical average.

Figure 1.19 shows the time histories of the fields when the wind is turned on impulsively.

[A page or so more discussion when this calculation is finished...]

1.6 — Retrospective/ prospective

This chapter has provided a brief overview of the elements comprising what we might call the standard kind of physical-biological model for oceanic problems. For pedagogical purposes, however, we avoided at each stage the complexities we expect to find in the real ocean. Such complexities (e.g., multiple components and species, different functional forms, downstream variation and eddies in the flow, ...) cannot be avoided if we wish to build a model which can more closely resemble observations and which can include information from laboratory experiments or fundamental limitations on biological processes. In the following chapters, we shall investigate many (but by no means all!) of the difficulties to understand how they arise and what impact they might have. Nevertheless, modelling efforts require a series of decisions about which elements/ processes to include or to exclude and how to represent the interactions among the different parts. We hope the remaining parts of this book will both encourage beginning scientists to think about modelling in new ways and remind experienced ones to be aware of and concerned about the effects their underlying assumptions may have on their results.