12.823 Problem Set 1

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Problem 1.

$$ln(\frac{P(t+15)}{P(t)}) = 15[P_h(t) - R(t) - G(t)], \text{ with}$$

$$P_h = \frac{pI_0}{kz_1}(1 - e^{-kz_1})(1 - N)(1 - V) \qquad R = R_0e^{rt} \qquad G = gZ$$

0.1 Baseline model run

To simplify the calculations, $P(t + 15)$ was rewritten as

$$P(t + 15) = P(t)e^{(15(P_h(t) - R(t) - G(t))}$$

At each time step past $t = 0$, $P(t + 15)$ was calculated and then iteratively used in the next time step. The baseline model run is plotted below, along with Riley's observations.

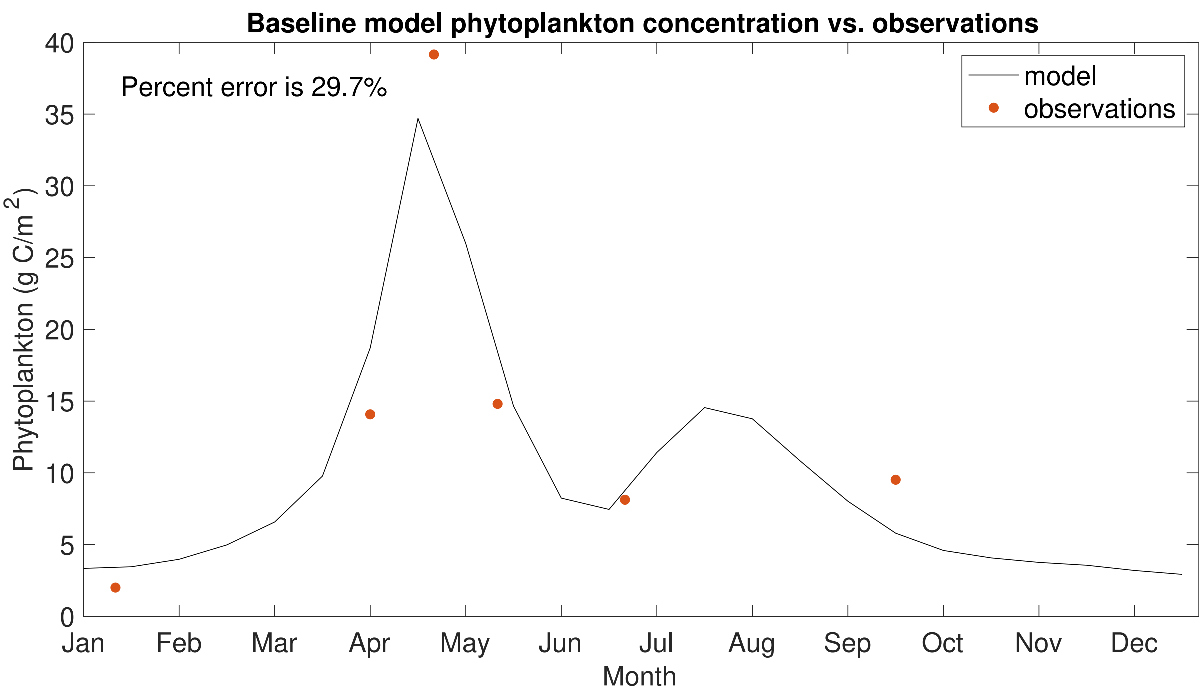

Figure 1: Baseline model run compared with observations.

The percent error here (and elsewhere in the problem set) is calculated as

$$\text{percent error} = 100 \times \text{abs}\left(\sum_{i=1}^{N_{obs}} \frac{P_{model}(t(i)) - P_{obs}(t_{obs}(i))}{P_{obs}(t_{obs}(i))}\right)/N_{obs}$$

i.e., the average percent error between the modeled phytoplankton concentration and the observed concentration. Because the model timestamps don't always align with the observation timestamps, the model output was interpolated with a spline function and the percent errors calculated from that interpolated data. The value of 29.7% calculated for the baseline case is of the same order as Riley's value of 27%, and the difference between the two may be attributed to carrying of significant figures or improved computating power leading to higher precision.

0.2 Sensitivity analysis

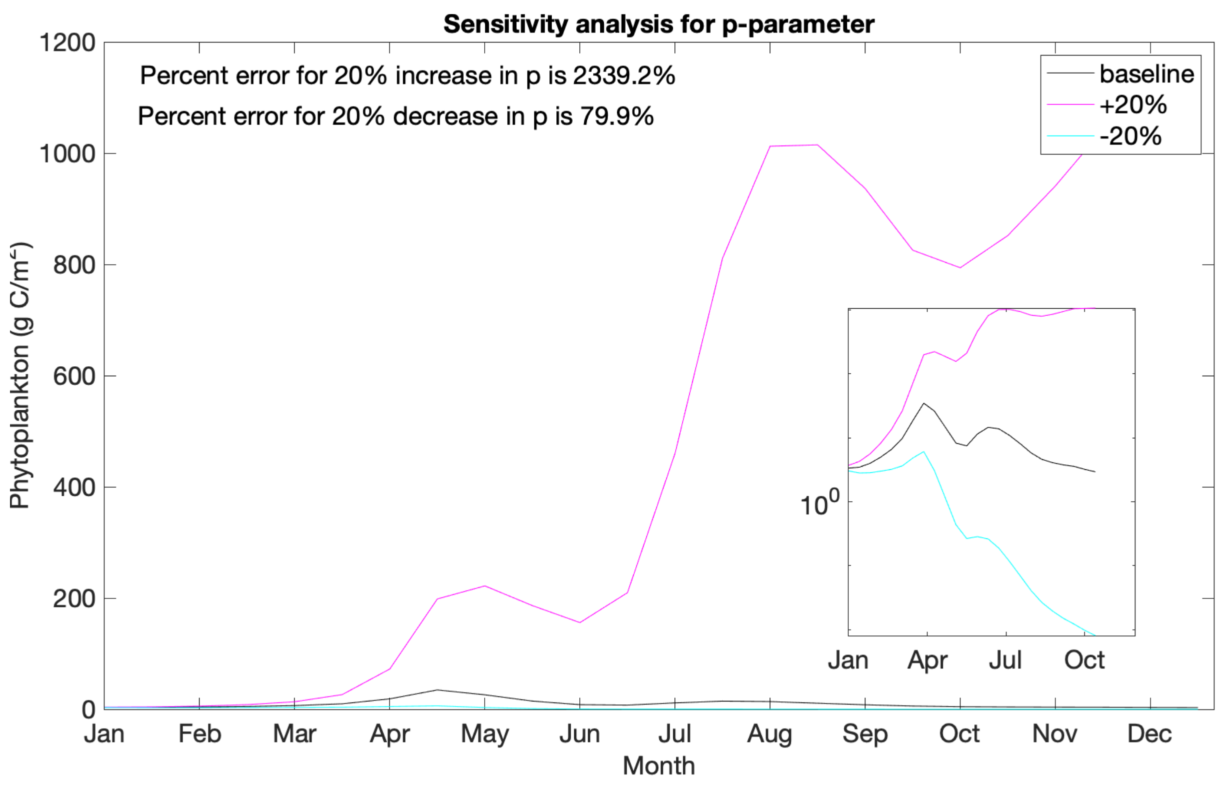

Figure 2: Sensitivity analysis on parameter p. The inset box shows the data plotted in log space.

p is the photosynthetic constant, i.e. how much photosynthesis (and growth) occurs for a given incident radiance I_0. Unsurprisingly, the model results are very sensitive to the value of this constant, with runaway growth for $1.2p = 3$, as the grazing and loss terms don't balance out the growth term even at low light levels. Similarly, $0.8p = 2$ leads to a damping out of the phytoplankton as early as April, with a small spring bloom before concentrations start to drop asymptotically toward zero.

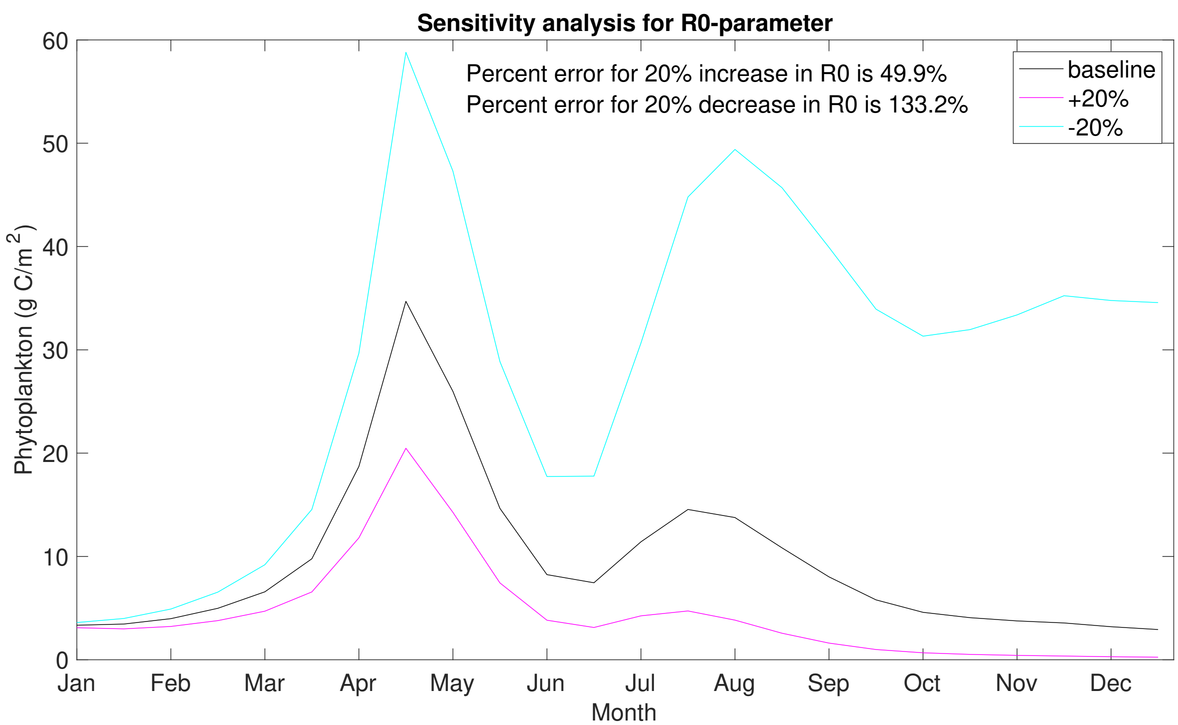

Figure 3: Sensitivity analysis on parameter R_0.

R_0 is the respiratory rate of phytoplankton at 0°C. It is less sensitive to a 20% increase or decrease than p, but a 20% decrease in R_0 does result in a second, almost equally large fall bloom and baseline levels close to 40 g C/m^2 at the end of the year. Again, it is intuitively reasonable that a reduction in the respiratory rate would lead to phytoplankton accumulation, as growth is dependent on the balance between photosynthesis and respiration. The asymmetric nature of the response for 20% increase and decrease may be related to exponential nature of the term it is multiplying.

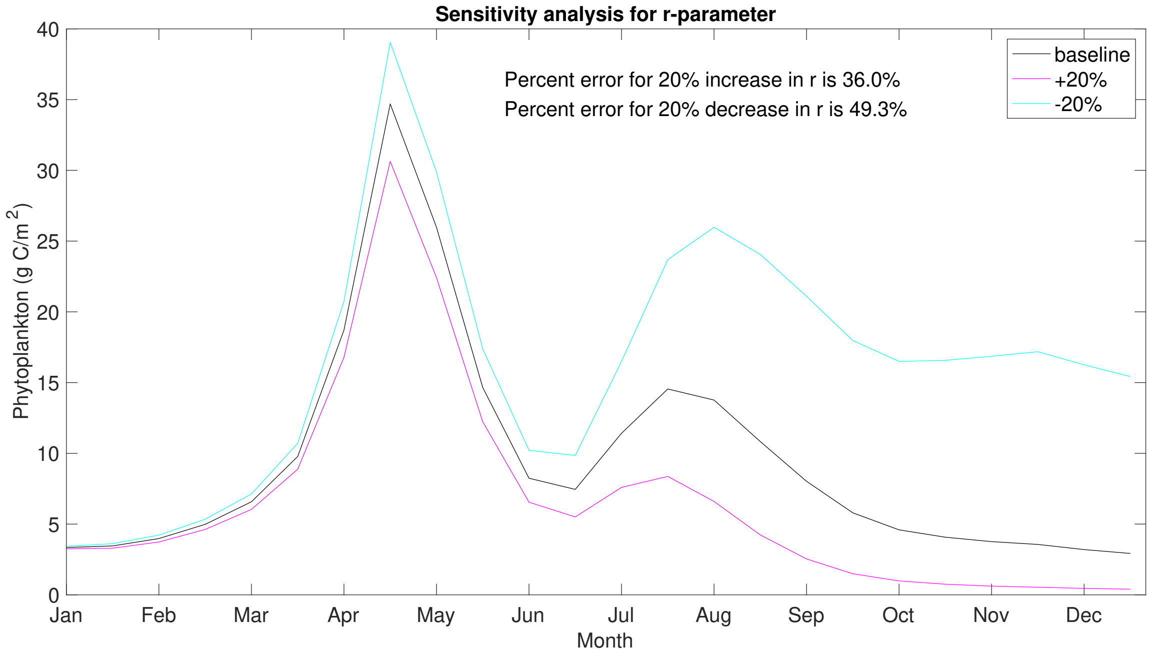

Figure 4: Sensitivity analysis on parameter r.

r relates the respiratory rate to the change in temperature. Interestingly, P is much less sensitive to changes in r than changes in R_0 or p. This might also be related to the exponential nature of e^{rT}, although I would have expected the opposite relationship, that the exponential term would amplify seasonal perturbations and lead to runaway growth or decay.

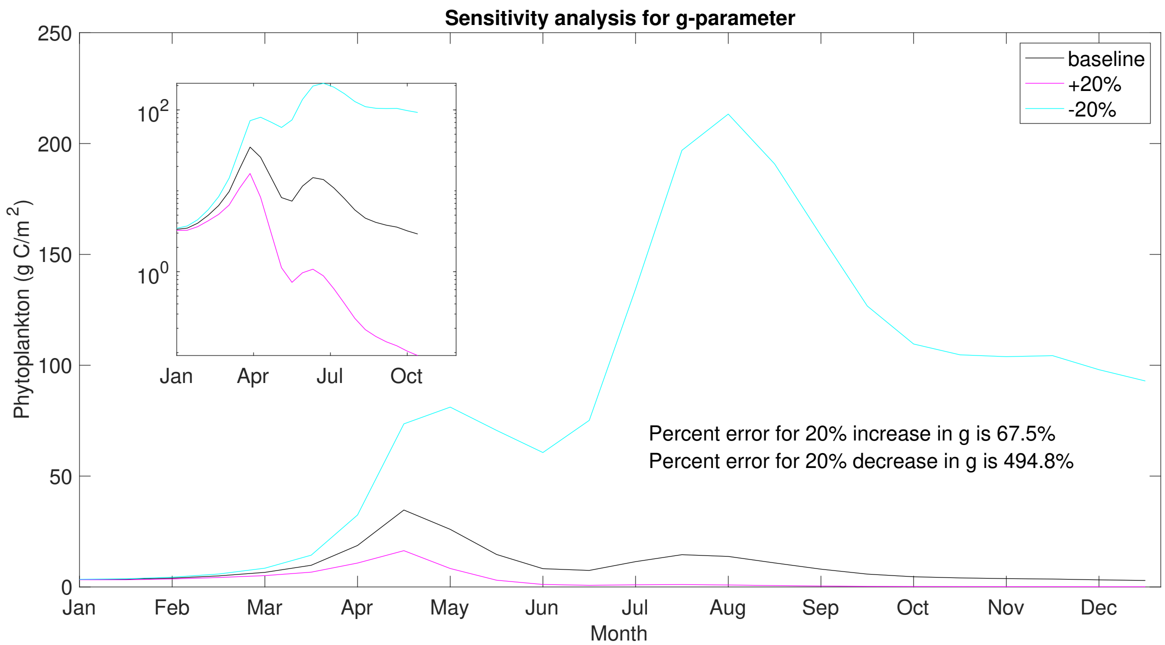

Figure 5: Sensitivity analysis on parameter g. The inset box shows the data plotted in log space.

g is the grazing constant by zooplankton. An increase in grazing leads to a decay in phytoplankton concentrations, unsurprisingly, and a decrease leads to elevated growth and the loss of periodicity. While the model is sensitive to the choice of g, interestingly, it is much more sensitive to p (see above). A choice of $0.8g$ doesn't lead to runaway growth in a 1-year time interval, suggesting that the fixed zooplankton concentrations (which are not independent in this model) act to prevent exponential growth even with a decrease in g. However, low enough g will eventually lead to runaway growth (plotted but not shown).

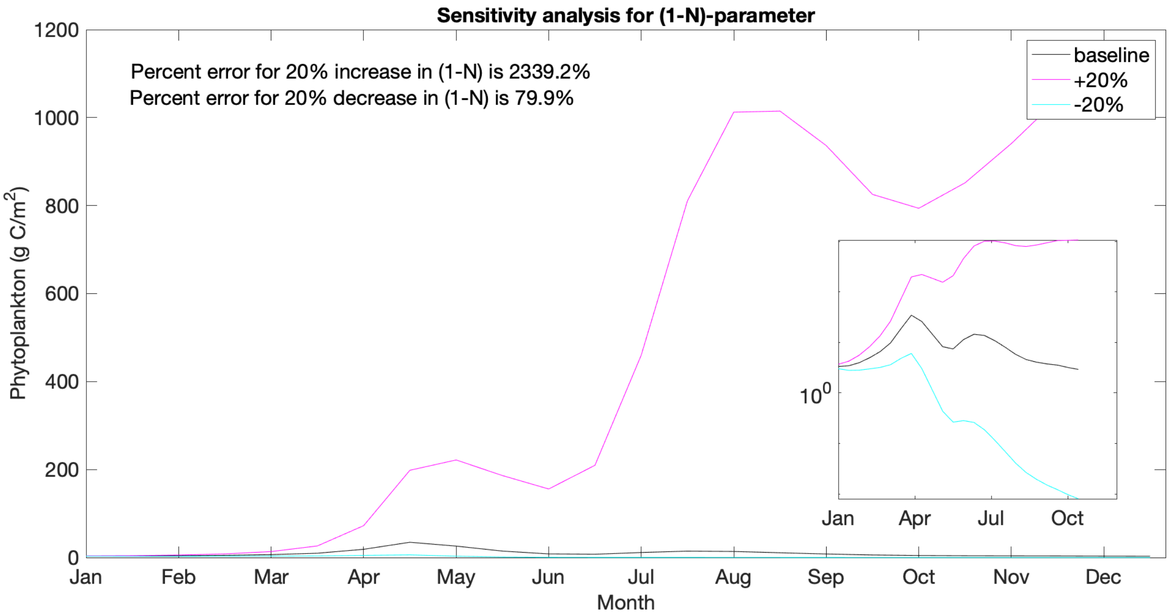

Figure 6: Sensitivity analysis on parameter $(1-N)$. The inset box shows the data plotted in log space.

The behavior on $(1-N)$ for the sensitivity is identical to the behavior for p. This is unsurprising, because the formulation for P_h in Riley's model is $P_h = \frac{pI_0}{kz_1}(1-e^{-kz_1})(1-N)(1-V)$, so both sensitivity analyses are equivalent to multiplying P_h by 1.2 or 0.8. From a more biological perspective, the $(1-N)$ term reflects nutrient limitation, and is only "turned on" when nutrients are low enough that the same amount of incident radiation will not lead to the same amount of growth. $1.2(1-N)$ actually reflects some sort of nutrient boost, where ample nutrients leads to a boost in growth.

Ultimately, the model is most sensitive to the p and $(1-N)$ parameters, less sensitive to g and less so to R_0, and fairly insensitive to the choice of r.

Errors for each sensitivity analysis are printed on the plot itself. While Riley's choice of parameters is based on literature review/experimental data, as opposed to an effort to fit the model to the data, they generally seem to yield a good fit — none of the sensitivity analyses improved the error estimate. After observing that the error for the r-parameter is quite close to my calculated error of 29.7% (which I am using instead of Riley's 27 % estimate — see above), I iterated through nearby values and found that $r = 0.0635$ yields an error of 27.8%, which is slightly (although perhaps not meaningfully) lower, perhaps because the lower respiration leads to a larger peak in the spring bloom and secondary fall bloom.

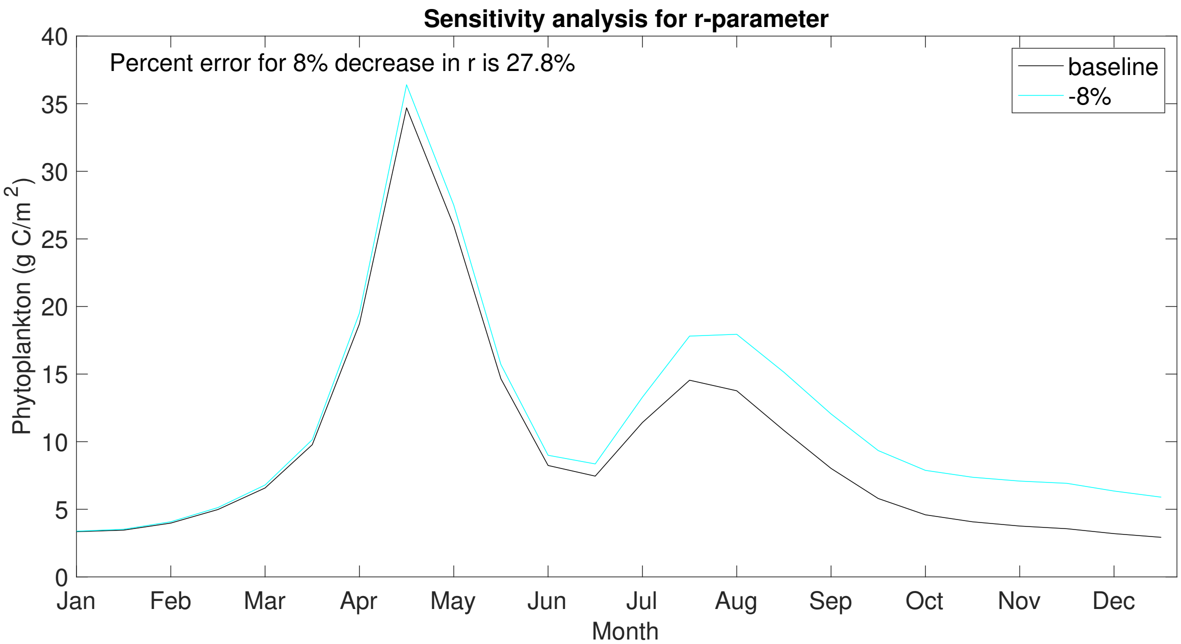

Figure 7: Output from using 92% of r as adjusted parameter.

0.3 Periodicity of P

I interpret the periodicity of P to imply that in a year interval, the starting and ending value are the same – i.e. P recurs over a period of 1 year. If we construct a longer time series, keeping all observations periodic (repeating annually, or every 360 days in this case), there is a grazing parameter g such that $P(360) = P(0)$, and thus the following year will have an identical pattern of P. The process of finding such a g is to figure out when $P(0) = P(360)$.

Earlier, we rewrote

$$ln(\frac{P(t+15)}{P(t)}) = 15[P_h(t) - R(t) - G(t)] \text{ as}$$
$$P(t+15) = P(t)e^{(15(P_h(t)-R(t)-G(t))}$$

Observing that

$$P(t) = P(t-15)e^{(15(P_h(t-15)-R(t-15)-G(t-15))}$$

This becomes

$$\begin{aligned} P(t+15) &= P(t-15)e^{(15(P_h(t-15)-R(t-15)-G(t-15))}e^{(15(P_h(t)-R(t)-G(t))} \\ &= P(t-15)e^{15(P_h(t)+P_h(t-15)+R(t)-(R(t)+R(t-15))+(G(t)+G(t-15)))} \end{aligned}$$

Using inductive reasoning, we can extend this back to t=0 by noting that P(t) is constructed by summing the forcing values at each time step and taking the exponential, i.e.

$$P(t) = P(0)\ \exp(\sum_{i=0}^{t-15} 15[P_h(i) - R(i) - G(i)]), \text{ where } i = 0, 15, 30, ..., t-15$$

This comes in handy when trying to find the value of g that makes P periodic (in that $P_0 = P(\text{end})$). We know P_0 and hence $P(t_{end})$, so this equation becomes

$$\begin{aligned}\frac{P(t_{end})}{P(0)} &= \exp(\sum_{i=0}^{t-15} 15[P_h(i) - R(i) - G(i)]) \\ 1 &= \exp(\sum_{i=0}^{t-15} 15[P_h(i) - R(i) - G(i)])\end{aligned}$$

Taking the natural log of both sides,

$$\begin{aligned}ln(1) = 0 &= \sum_{i=0}^{t-15} 15[P_h(i) - R(i) - g * Z(i)] \\ g &= \sum_{i=0}^{t-15}[P_h(i) - R(i)]/\sum_{i=0}^{t-15} Z(i)]\end{aligned}$$

When we plug in the actual data, we get a value of 0.00743 for g, which is extremely close to the original value given for g.

This analysis was repeated for 20 % increases and decreases in all of the parameters used for the sensitivity analysis — a table of the results is shown below. A plot of what these periodic concentrations look like is also shown.

parameter	20% increase	20% decrease
p	0.01	0.0049
R_0	0.0064	0.0085
r	0.0065	0.0082

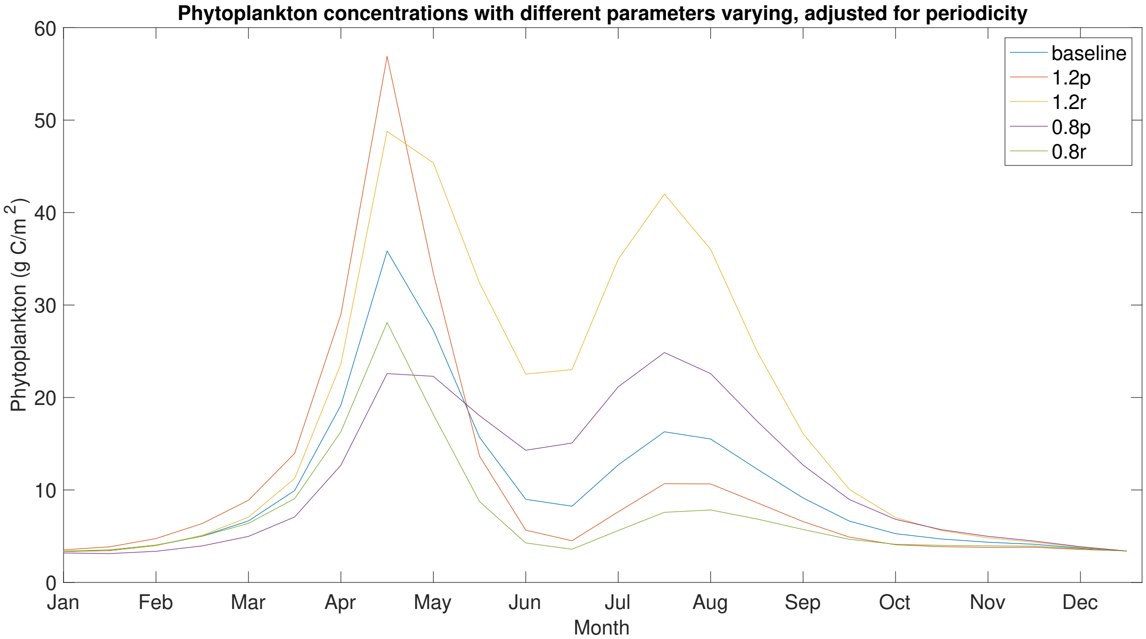

Figure 8: Plots of periodic P, which is dependent on grazing parameter.

0.4 10-year time series

The time series is plotted below, with both (randomized) zooplankton concentrations and phytoplankton concentrations.

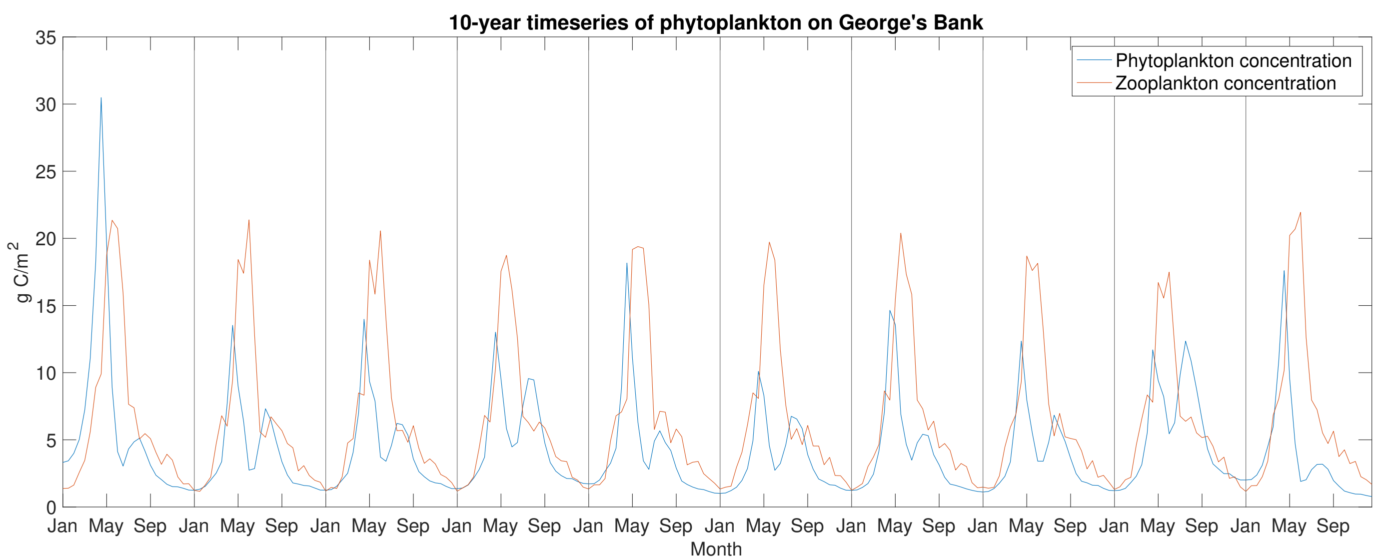

Figure 9: 10-year time-series of phytoplankton concentrations, along with randomized zooplankton concentrations.

The 10-year time series reflects the pressure that grazing zooplankton can put on phytoplankton, even when all other parameters are held constant. In this time series, P is initially higher than Z in the first year, but in the second year, Z starts higher and peaks earlier as a function of this 20% random property. This increase in the gZ term decreases P, and because $P(t+15)$ is a multiple of $P(t)$ with some exponential factor, decreased P propagates forward in time, reducing the average amplitude of the spring peaks. Overall, P lags Z quite closely, such as in the last year of the time series, where a higher than average series of Z values in the summer leads to a much smaller fall peak.

Interestingly, both the spring and fall bloom show up in most years of the timeseries, albeit with different magnitudes. Looking at a plot of the seasonal cycle of the observations and P from the baseline model, it's clear that the second peak is largely tied to a decline in zooplankton concentrations while nutrients and light are still readily available, which decreases the grazing term. Hence, because Z only varies within $\pm$ 20%, the two blooms persist in the longer time-series. However, because Riley's approach treats Z as independent from P, rather than adding a $\frac{\partial Z}{\partial t}$ term, we don't see the response of the Z term to the P term. In other words, the response of P to increased grazing is apparent in the time-series, but zooplankton populations don't decrease as the source of food, P, decreases.

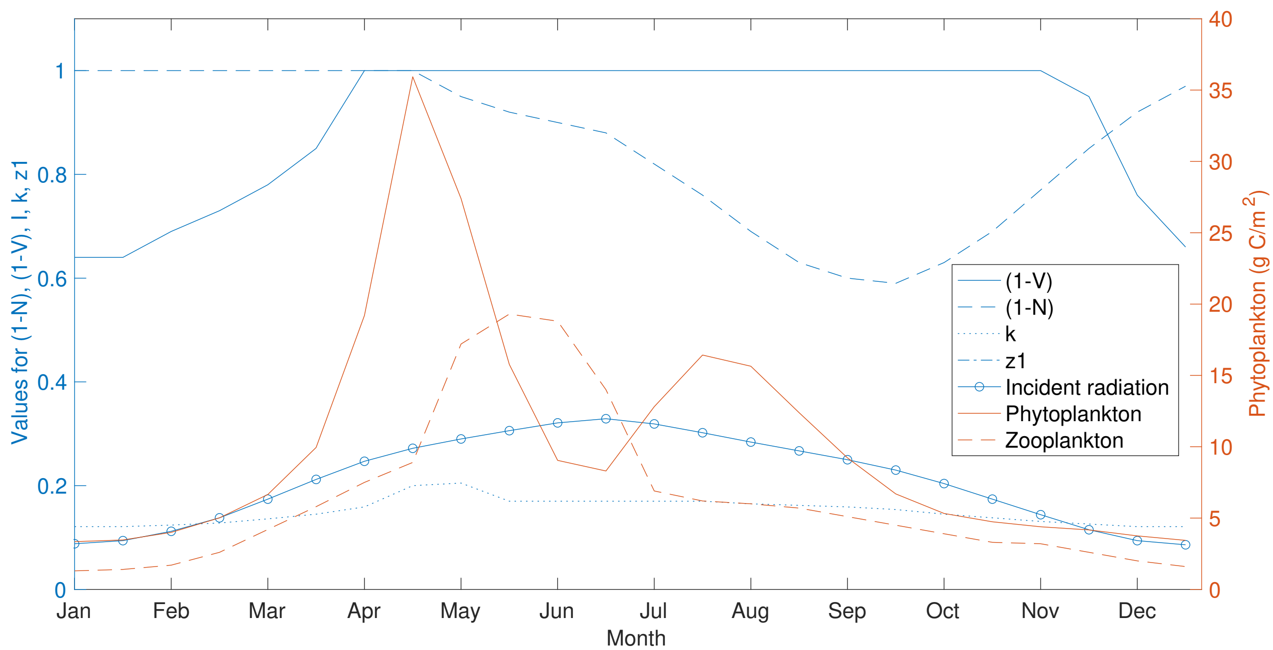

Figure 10: Seasonal cycle of P, Z, z_1, k, I, (1-N), (1-V), and respiration R